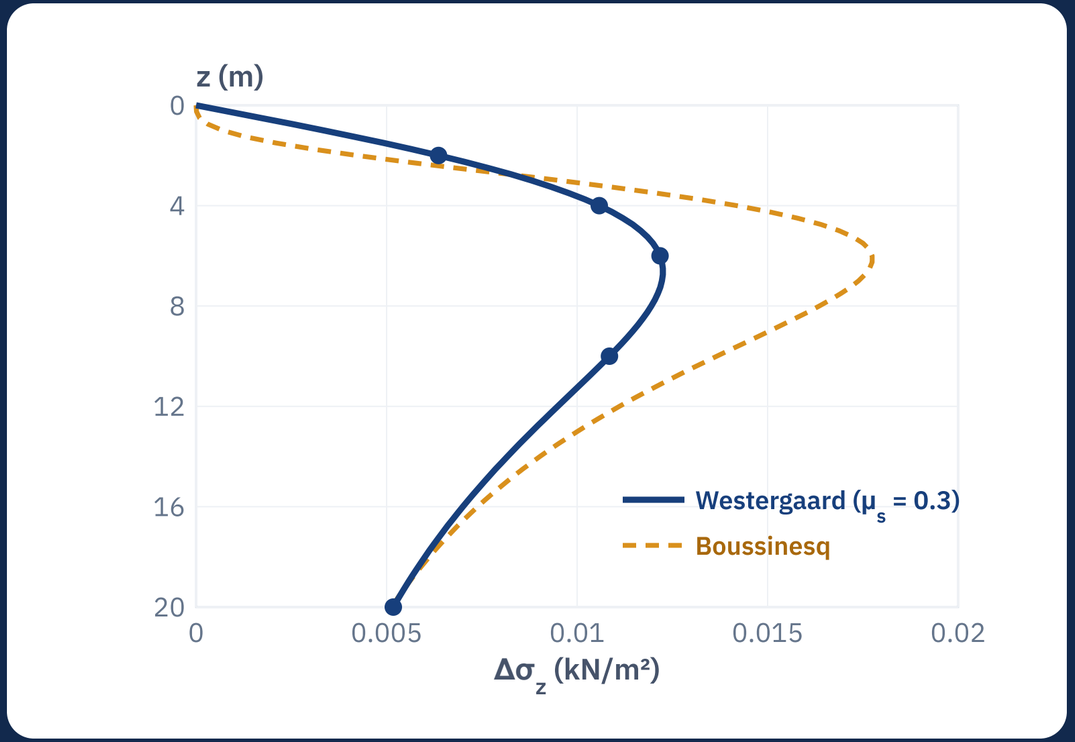
Geotechnical lecture presentation · Stresses in the soil mass

Westergaard: stress under a point load

Lecture: stress under a point load in a layered soil whose lateral strains are confined by rigid plates; one point is solved by hand from start to finish.

- Westergaard medium
- η and I_5
- effect of μ_s
- comparison with Boussinesq
- Superposition



Introduction

What will we learn in this lecture?

The **Westergaard medium**: an elastic soil idealisation containing thin, rigid horizontal plates that do not permit lateral deformation

Vertical stress increase under a point load: **η**, **influence value I_5** and **$\Delta\sigma_z = (P/z^2)\, I_5$**

Where the Westergaard stress is smaller and where larger than the **Boussinesq** stress; the role of Poisson's ratio **μ_s**

Variation with depth and horizontal distance, the depth of maximum stress

Superposition of multiple point loads

LECTURE FLOW

Each topic is covered in **three steps**:

1 · **Theory**: formula and rationale

2 · **Problem statement**: given and required

3 · **Solution**: step by step, with numbers

Example: P = 5 kN, point A is at r = 5 m from the load and z = 4 m deep; in the soil between the plates μ_s = 0.3.

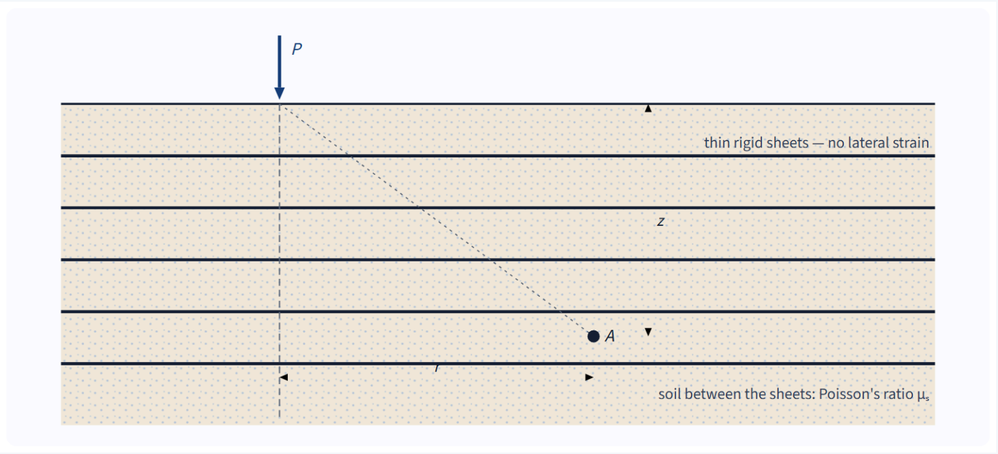
Soil with a rigid horizontal plate

THEORY

Boussinesq-based solutions rely on integrating the point-load solution for a **homogeneous and isotropic** half-space. Many soils, however, are **layered**

Example: a clay containing fine sand seams and whose horizontal deformation is constrained by stiffer layers

Westergaard (1938) idealised this soil as follows: an elastic medium containing a large number of **thin, rigid horizontal sheets** that do not permit lateral deformation



Vertical stress increase and η

THEORY

The increase in vertical stress caused by the point load **P** at the surface at point A, at horizontal distance **r** and depth **z**; μ_s is the Poisson's ratio of the soil between rigid plates, $r = \sqrt{(x^2 + y^2)}$

The solution is rewritten with an **influence value** as in Boussinesq I_1

$$\Delta\sigma_z = \frac{P\,\eta}{2\pi z^2}\left[\frac{1}{\eta^2 + (r/z)^2}\right]^{3/2}$$

$$\eta = \sqrt{\frac{1 - 2\mu_s}{2 - 2\mu_s}}$$

$\Delta\sigma_z = (P/z^2)\, I_5$

THEORY

The Westergaard solution is written in the form **$\Delta\sigma_z = (P/z^2)\, I_5$**, as in Boussinesq; I_5 depends only on **$r/(\eta z)$** and **η**

For comparison, the Boussinesq solution: $\Delta\sigma_z = (P/z^2)\, I_1$, where I_1 depends only on r/z (the vertical stress is independent of Poisson's ratio)

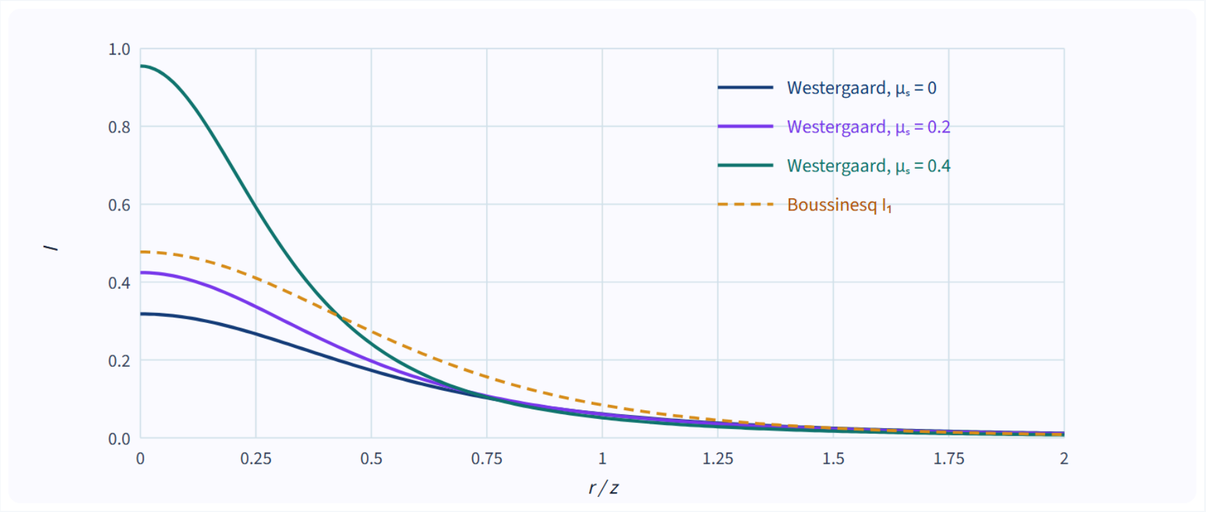
$$\Delta\sigma_z = \left(\frac{P}{z^2}\right) I_5 \qquad I_5 = \frac{1}{2\pi\eta^2}\left[\left(\frac{r}{\eta z}\right)^2 + 1\right]^{-3/2}$$

$$\text{Boussinesq:} \quad I_1 = \frac{3}{2\pi}\left[\left(\frac{r}{z}\right)^2 + 1\right]^{-5/2}$$

Variation of the influence value with r/z

THEORY

r/z	0	0.1	0.2	0.3	0.4	0.5	0.6	0.8	1	1.5	2	3	5
$\mu_s = 0$	0.3183	0.3090	0.2836	0.2483	0.2099	0.1733	0.1411	0.0925	0.0613	0.0247	0.0118	0.0038	0.0009
$\mu_s = 0.2$	0.4244	0.4080	0.3646	0.3074	0.2491	0.1972	0.1547	0.0953	0.0604	0.0229	0.0107	0.0034	0.0008
$\mu_s = 0.4$	0.9549	0.8750	0.6916	0.4997	0.3480	0.2416	0.1700	0.0897	0.0516	0.0173	0.0076	0.0023	0.0005
Boussinesq I_1	0.4775	0.4657	0.4329	0.3849	0.3295	0.2733	0.2214	0.1386	0.0844	0.0251	0.0085	0.0015	0.0001



Westergaard and Boussinesq beneath the load

THEORY

Directly beneath the load (r = 0), the Westergaard stress is **lower** than the Boussinesq stress for **μ_s = 0**: $I_5 = 1/\pi = 0.3183$; for Boussinesq, $I_1 = 3/2\pi = 0.4775$. Rigid slabs spread the load laterally

Beneath the load the stress **increases as μ_s increases**; at **μ_s = 0.5**, η = 0 and the solution **loses its validity**. The module accepts μ_s between 0 and 0.49

μ_s	0.00	0.10	0.20	0.25	0.30	0.40	0.45	0.49
η	0.7071	0.6667	0.6124	0.5774	0.5345	0.4082	0.3015	0.1400
η^2	0.5000	0.4444	0.3750	0.3333	0.2857	0.1667	0.0909	0.0196
I_5 (r = 0) = $1/(2\pi\eta^2)$	0.3183	0.3581	0.4244	0.4775	0.5570	0.9549	1.7507	8.1169

Superposition

THEORY

Since the Westergaard medium is **elastic**, the stress from each load is found using its **own distance r** to point A, and the stresses are **summed**

Loads are located at their own positions on the ground; up to eight loads can be entered and no two may be at the same position

The Boussinesq comparison value is also summed in the same way

$$\Delta\sigma_z = \sum_k \frac{P_k}{z^2}\, I_{5,k} \qquad r_k = \sqrt{(x - x_k)^2 + (y - y_k)^2} \qquad I_{5,k} = I_5\left(\frac{r_k}{\eta z}\right)$$

Problem statement

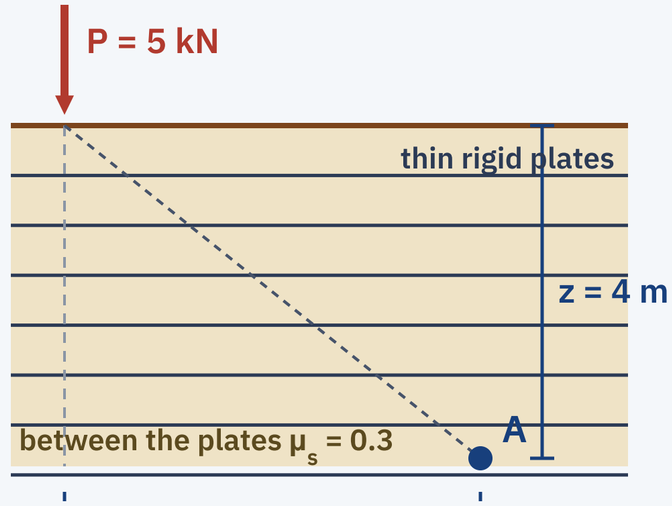
EXAMPLE

Given

Load	**at surface** $P = 5$ **kN,** $(x_L;\, y_L) = (0;\, 0)$
Point A	$x = 3$ **m,** $y = 4$ **m,** $z = 4$ **m**
Soil	**between plates** $\mu_s = 0.3$

REQUIRED

η, **r/(ηz)**, influence value $\mathbf{I_5}$, $\mathbf{\Delta\sigma_z}$ and comparison with the Boussinesq stress.



Solution (1/3)

SOLUTION

1 Westergaard constant η

$$\eta = \sqrt{\frac{1-2\mu_s}{2-2\mu_s}} = \sqrt{\frac{1-0.6}{2-0.6}} = \sqrt{0.2857} = \mathbf{0.5345}$$

2 Horizontal distance and r/(ηz)

$$r = \sqrt{x^2+y^2} = \sqrt{3^2+4^2} = \mathbf{5.00\ m}$$

$$\frac{r}{\eta z} = \frac{5}{0.5345\,(4)} = \mathbf{2.3385}$$

η = **0.5345** (η² = 0.2857). Point A is at a distance **r = 5 m** from the load; in Boussinesq this distance would enter as r/z = 1.25, in Westergaard it enters as r/(ηz) = **2.3385**.

Solution (2/3)

SOLUTION

3 Influence value I_5

$$I_5 = \frac{1}{2\pi\eta^2}\left[\left(\frac{r}{\eta z}\right)^2 + 1\right]^{-3/2} = \frac{1}{2\pi\,(0.2857)}\left[2.3385^2 + 1\right]^{-3/2}$$

$$= 0.5570 \times [6.4687]^{-3/2} = 0.5570 \times 0.06078 = \mathbf{0.03386}$$

4 Vertical stress increase

$$\Delta\sigma_z = \frac{P}{z^2}\,I_5 = \frac{5}{4^2}\,(0.03386) = 0.3125 \times 0.03386 = \mathbf{0.01058}\ \mathrm{kN/m^2}$$

The 5 kN point load at the surface produces an increase of **$\Delta\sigma_z$ = 0.01058 kN/m²** at point A in the Westergaard medium (I_5 = 0.03386).

Solution (3/3)

SOLUTION

5 Check with the first form of the equation

$$\Delta\sigma_z = \frac{P\eta}{2\pi z^2}\left[\frac{1}{\eta^2 + (r/z)^2}\right]^{3/2} = \frac{5\,(0.5345)}{2\pi\,(16)}\left[\frac{1}{0.2857 + 1.5625}\right]^{3/2} = \mathbf{0.01058}$$

6 Boussinesq (comparison)

$$I_1 = \frac{3}{2\pi}\left[\left(\frac{r}{z}\right)^2 + 1\right]^{-5/2} = \frac{3}{2\pi}\,[2.5625]^{-5/2} = \mathbf{0.04542}$$

$$\Delta\sigma_{z,B} = \frac{5}{4^2}\,(0.04542) = \mathbf{0.01419}\ \mathrm{kN/m^2} \qquad \frac{\Delta\sigma_z}{\Delta\sigma_{z,B}} = 0.745$$

The two formulations give the same result. At this point the Westergaard stress is about **74.5%** of the Boussinesq stress: because rigid plates spread the load laterally, the stress at intermediate depth is smaller.

Check the solution in GeoTerva ↗

Problem statement

EXAMPLE

Given

Load and point	$P = 5$ **kN; point A is** $r = 5$ **m from the load;** $\mu_s = 0.3$
Depths	$z = 2,\ 4,\ 6 \ldots 20$ **m (the module's profile table)**

REQUIRED

At every depth: **r/(ηz)**, $\mathbf{I_5}$, Westergaard and Boussinesq stresses; and the **depth of the maximum stress**.

Solution (1/2)

SOLUTION

z (m)	r/(ηz)	I_5	$\Delta\sigma_z$ Westergaard	$\Delta\sigma_z$ Boussinesq	Ratio W/B
2	4.677	0.0051	**0.00636**	0.00422	1.509
4	2.339	0.0339	**0.01058**	0.01419	0.745
6	1.559	0.0877	**0.01218**	0.01774	0.686
8	1.169	0.1529	**0.01195**	0.01636	0.731
10	0.935	0.2170	**0.01085**	0.01367	0.794
12	0.780	0.2733	**0.00949**	0.01111	0.854
14	0.668	0.3202	**0.00817**	0.00902	0.905
16	0.585	0.3584	**0.00700**	0.00739	0.948
18	0.520	0.3892	**0.00601**	0.00612	0.982
20	0.468	0.4140	**0.00518**	0.00513	1.009

At **z = 2 m** the Westergaard stress is larger (0.00636 and 0.00422); **at intermediate depths it is smaller** (z = 6 m: 0.01218 and 0.01774); at **z = 20 m** the two solutions approach each other (0.00518 and 0.00513).

Solution (2/2)

SOLUTION

1 by hand for z = 10 m

$$\frac{r}{\eta z} = \frac{5}{0.5345\,(10)} = 0.9354$$

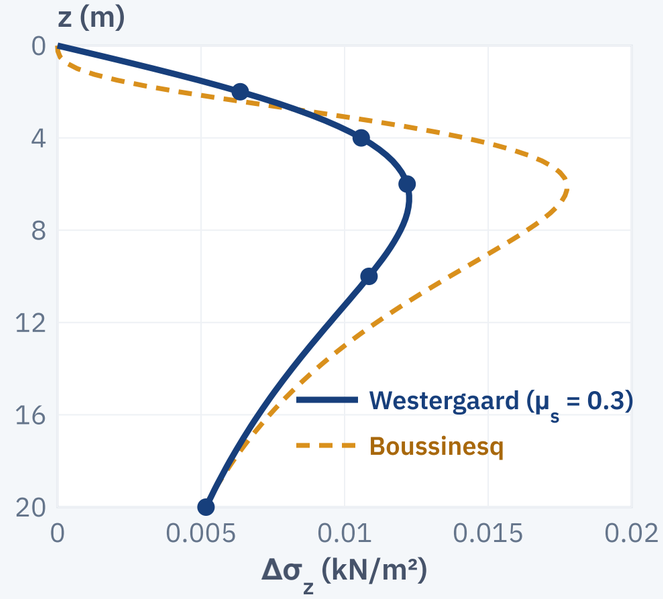
$$I_5 = 0.5570\,[1.8750]^{-3/2} = 0.2170$$

$$\Delta\sigma_z = \frac{5}{10^2}\,(0.2170) = \mathbf{0.01085}\ \mathrm{kN/m^2}$$

2 Depth of maximum stress ($d\Delta\sigma_z/dz = 0$)

$$\Delta\sigma_z = \frac{P\eta}{2\pi}\,z\,(r^2 + \eta^2 z^2)^{-3/2}$$

$$z^* = \frac{r}{\sqrt{2}\,\eta} = \frac{5}{1.4142\,(0.5345)} = \mathbf{6.61}\ \mathrm{m}$$



Maximum stress: Westergaard at **z = 6.61 m** (0.01225), Boussinesq (z* = r√1.5) at **z = 6.12 m** (0.01775 kN/m²).

Check the solution in GeoTerva ↗

Problem statement

EXAMPLE

Given

Load and point	$P = 5$ **kN,** $r = 5$ **m,** $z = 4$ **m (point in Example 1)**
Varying	$\mu_s = 0;\ 0.1;\ 0.2;\ 0.3;\ 0.4;\ 0.45;\ 0.49$

REQUIRED

For each μ_s, **η**, **r/(ηz)**, $\mathbf{I_5}$, $\mathbf{\Delta\sigma_z}$ and the ratio to Boussinesq.

Solution (1/3)

SOLUTION

μ_s	η	$r/(\eta z)$	I_5	$\Delta\sigma_z$ (kN/m²)	$\Delta\sigma_z$ / Boussinesq
0.00	0.7071	1.768	0.03799	**0.01187**	0.836
0.10	0.6667	1.875	0.03732	**0.01166**	0.822
0.20	0.6124	2.041	0.03614	**0.01129**	0.796
0.30	0.5345	2.339	0.03386	**0.01058**	0.745
0.40	0.4082	3.062	0.02858	**0.00893**	0.629
0.45	0.3015	4.146	0.02257	**0.00705**	0.497
0.49	0.1400	8.927	0.01120	**0.00350**	0.247

At this point (r/z = 1.25), **as μ_s increases, the Westergaard stress decreases**: 0.01187 at μ_s = 0 and 0.00350 kN/m² at μ_s = 0.49. On the load axis the opposite occurs (Example 4): the direction of the effect depends on the position of the point.

Solution (2/3)

SOLUTION

1 $\mu_s = 0$

$$\eta^2 = \frac{1}{2} = 0.5000\,, \quad \eta = 0.7071\,, \quad \frac{r}{\eta z} = \frac{5}{0.7071(4)} = 1.7678$$

$$I_5 = \frac{1}{\pi}\left[4.1250\right]^{-3/2} = 0.3183 \times 0.11936 = 0.03799$$

$$\Delta\sigma_z = \frac{5}{16}\,(0.03799) = \mathbf{0.01187}$$

When Poisson's ratio is zero, η = 1/√2 = 0.7071; at this point **$\Delta\sigma_z$ = 0.01187 kN/m²** is found (83.6% of Boussinesq).

Solution (3/3)

SOLUTION

2 $\mu_s = 0.4$

$$\eta^2 = \frac{1 - 0.8}{2 - 0.8} = 0.1667\,, \quad \eta = 0.4082\,, \quad \frac{r}{\eta z} = \frac{5}{0.4082(4)} = 3.0619$$

$$I_5 = 0.9549\,[10.3750]^{-3/2} = 0.9549 \times 0.02992 = 0.02858$$

$$\Delta\sigma_z = \frac{5}{16}\,(0.02858) = \mathbf{0.00893}$$

As the Poisson's ratio of the soil between the plates increases, η decreases; r/(ηz) increases and I_5 at this point decreases: **0.01187 → 0.00893 kN/m²** (μ_s = 0 → 0.4).

Check the solution in GeoTerva ↗

Problem statement

EXAMPLE

Given

Load	$P = 5$ **kN,** $\mu_s = 0.3$
Depth	$z = 4$ **m; horizontal distance** $r = 0,\ 1,\ 2 \ldots 10$ **m**
Problem statement	**how do the Westergaard and Boussinesq stresses compare on the load axis; for what value of** μ_s **are the two solutions equal?**

REQUIRED

$\Delta\sigma_z$ under the load (r = 0), the same value for μ_s = 0 and the **horizontal profile table**.

Solution (1/3)

SOLUTION

1 r = 0 for $\mu_s = 0.3$

$$I_5(0) = \frac{1}{2\pi\eta^2} = \frac{1}{2\pi\,(0.2857)} = 0.5570 \qquad \Delta\sigma_z = \frac{5}{16}\,(0.5570) = \mathbf{0.1741}$$

$$\text{Boussinesq: } I_1(0) = \frac{3}{2\pi} = 0.4775 \;\Rightarrow\; \Delta\sigma_{z,B} = \frac{5}{16}\,(0.4775) = \mathbf{0.1492}$$

2 For $\mu_s = 0$, r = 0

$$I_5(0) = \frac{1}{\pi} = 0.3183 \;\Rightarrow\; \Delta\sigma_z = \frac{5}{16}\,(0.3183) = \mathbf{0.0995} \qquad \frac{\Delta\sigma_z}{\Delta\sigma_{z,B}} = \frac{1/\pi}{3/2\pi} = \frac{2}{3}$$

μ_s = 0: Westergaard is **2/3** of Boussinesq (0.0995 and 0.1492); **μ_s = 0.3**: it is larger than it (0.1741); the two solutions become equal at **μ_s = 0.25**.

Solution (2/3)

SOLUTION

3 μ_s at which the two solutions are equal on the axis

$$\frac{1}{2\pi\eta^2} = \frac{3}{2\pi} \Rightarrow \eta^2 = \frac{1}{3} = \frac{1-2\mu_s}{2-2\mu_s} \Rightarrow \mu_s = \mathbf{0.25}$$

4 Ratio on the axis: Westergaard / Boussinesq = $(1/2\pi\eta^2) / (3/2\pi) = 1/(3\eta^2)$

μ_s	0.00	0.10	0.20	0.25	0.30	0.40	0.45	0.49
η^2	0.5000	0.4444	0.3750	0.3333	0.2857	0.1667	0.0909	0.0196
$\Delta\sigma_z$ Westergaard / Boussinesq	0.667	0.750	0.889	1.000	1.167	2.000	3.667	17.000

On the load axis the ratio grows rapidly with μ_s: 2/3 at $\mu_s = 0$, 1 at $\mu_s = 0.25$, 3.67 at $\mu_s = 0.45$; as μ_s approaches 0.5, $\eta \to 0$ and the ratio increases without bound.

Solution (3/3)

SOLUTION

r (m)	r/z	r/(ηz)	I_5	$\Delta\sigma_z$ Westergaard	$\Delta\sigma_z$ Boussinesq	Ratio W/B
0	0.00	0.000	0.5570	**0.17408**	0.14921	1.167
1	0.25	0.468	0.4140	**0.12938**	0.12822	1.009
2	0.50	0.935	0.2170	**0.06780**	0.08541	0.794
3	0.75	1.403	0.1089	**0.03403**	0.04889	0.696
4	1.00	1.871	0.0584	**0.01824**	0.02638	0.691
5	1.25	2.339	0.0339	**0.01058**	0.01419	0.745
6	1.50	2.806	0.0211	**0.00658**	0.00784	0.840
8	2.00	3.742	0.0096	**0.00300**	0.00267	1.123
10	2.50	4.677	0.0051	**0.00159**	0.00105	1.509

The Westergaard stress is above that of Boussinesq on the axis; after **r/z ≈ 0.26** it falls below Boussinesq (r = 2 m: 0.06780 and 0.08541) and after **r/z ≈ 1.81** (r ≈ 7.2 m) it rises above it again (r = 10 m: 0.00159 and 0.00105): at a distance the Westergaard stress decreases more slowly.

Check the solution in GeoTerva ↗

Problem statement

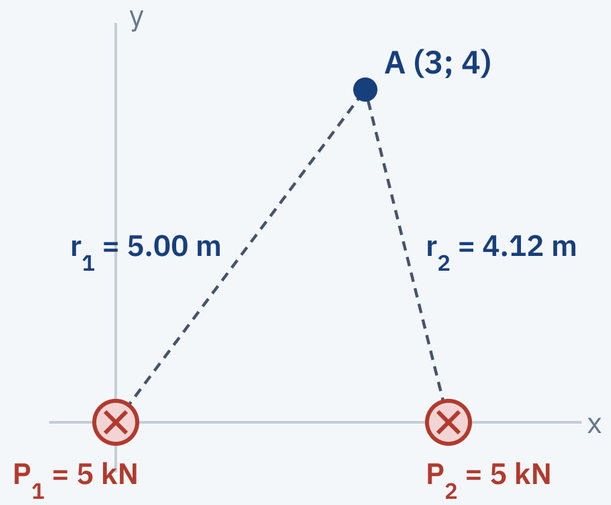
EXAMPLE

REQUIRED

The **r**, $\mathbf{I_5}$, $\mathbf{\Delta\sigma_z}$ values of each load; total stress and the Boussinesq sum.

Given

Loads	$P_1 = 5$ **kN,** $(0;\ 0)$ **and** $P_2 = 5$ **kN,** $(4;\ 0)$
Point A	$x = 3$ **m,** $y = 4$ **m,** $z = 4$ **m**
Soil	$\mu_s = 0.3$



Solution (1/2)

SOLUTION

1 P_1 (0; 0)

$$r_1 = \sqrt{3^2 + 4^2} = 5.000 \text{ m}$$

$$\frac{r_1}{\eta z} = \frac{5.000}{0.5345(4)} = 2.3385\,, \quad I_5 = 0.03386$$

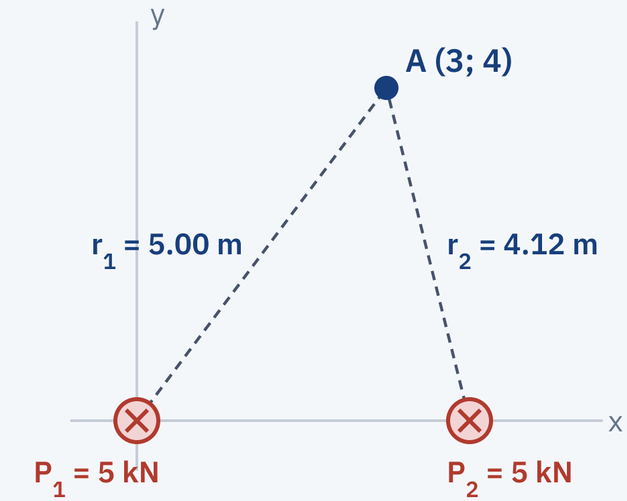
$$\Delta\sigma_{z(1)} = \frac{5}{16}\,(0.03386) = \mathbf{0.01058}\ \text{kN/m}^2$$

2 P_2 (4; 0)

$$r_2 = \sqrt{(3-4)^2 + 4^2} = 4.123 \text{ m}$$

$$\frac{r_2}{\eta z} = \frac{4.123}{0.5345(4)} = 1.9284\,, \quad I_5 = 0.05434$$

$$\Delta\sigma_{z(2)} = \frac{5}{16}\,(0.05434) = \mathbf{0.01698}\ \text{kN/m}^2$$



Load 2 is closer: **0.01698**; load 1: **0.01058 kN/m²**.

Solution (2/2)

SOLUTION

1 Stresses are summed

$$\Delta\sigma_z = \Delta\sigma_{z(1)} + \Delta\sigma_{z(2)} = 0.01058 + 0.01698 = \mathbf{0.02756}\ \mathrm{kN/m^2}$$

$$\Delta\sigma_{z,B} = 0.01419 + 0.02442 = 0.03862\ \mathrm{kN/m^2}$$

2 Depth profile beneath the same point (sum of the two loads)

z (m)	2	4	6	8	10	12	16	20
$\Delta\sigma_z$ Westergaard	0.0174	0.0276	0.0301	0.0282	0.0247	0.0210	0.0150	0.0108
$\Delta\sigma_z$ Boussinesq	0.0137	0.0386	0.0430	0.0371	0.0298	0.0237	0.0153	0.0105

At point A (z = 4 m) the two loads together produce **$\Delta\sigma_z$ = 0.02756 kN/m²**; the Boussinesq sum is 0.03862 kN/m². The total is the sum of the values found for each load individually; the value of a single load (Example 1) is a part of this total.

Check the solution in GeoTerva ↗

Summary

Key points

Westergaard medium: thin rigid horizontal plates prevent lateral deformation; $\mathbf{\Delta\sigma_z = (P/z^2)\, I_5}$, $I_5 = [1/(2\pi\eta^2)]\,[(r/\eta z)^2 + 1]^{-3/2}$, $\eta = \sqrt{[(1 - 2\mu_s)/(2 - 2\mu_s)]}$

Example (P = 5 kN, r = 5 m, z = 4 m, μ_s = 0.3): η 0.5345; r/(ηz) 2.339; I_5 0.03386 → **$\Delta\sigma_z$ = 0.01058 kN/m²**; Boussinesq 0.01419

On the load axis $I_5 = 1/(2\pi\eta^2)$: for $\mu_s = 0$, $1/\pi = 0.3183$ (Boussinesq $3/2\pi = 0.4775$); the stress increases with μ_s, and at $\mu_s = 0.5$ the solution loses its validity

Depth: at intermediate depths the Westergaard stress is smaller, and at great depth the two solutions approach each other; the maximum stress occurs at $z = r/(\sqrt{2}\,\eta) = 6.61$ m (Boussinesq: $r\sqrt{1.5} = 6.12$ m)

Superposition: each load is taken with its own distance r, and the stresses are added (two loads in the example: 0.02756 kN/m²)

References

Das, B.M. (2018). Principles of Geotechnical Engineering, 9th ed. Cengage Learning.

Westergaard, H.M. (1938). A problem of elasticity suggested by a problem in soil mechanics: soft material reinforced by numerous strong horizontal sheets. In Contributions to the Mechanics of Solids, Stephen Timoshenko 60th Anniversary Volume. Macmillan, New York, 268–277.

NOTE

All the numerical examples have been produced with the module's own calculation engine; the module gives the same values. The module's opening example is P = 5 kN, r = 5 m, μ_s = 0.3 and z = 4 m.

GeoTerva

Online geotechnical analysis

From theory to practice

Enter your own individual loads, point A and the μ_s value in GeoTerva; the module provides η, the influence value I_5, each load's contribution, the comparison with Boussinesq, the depth profile, the step-by-step solution and the report together.

Open the module → | **Theory page →**

Open on your phone
The example modules need no registration.

If you would like to use them in your lectures: info@geoterva.com