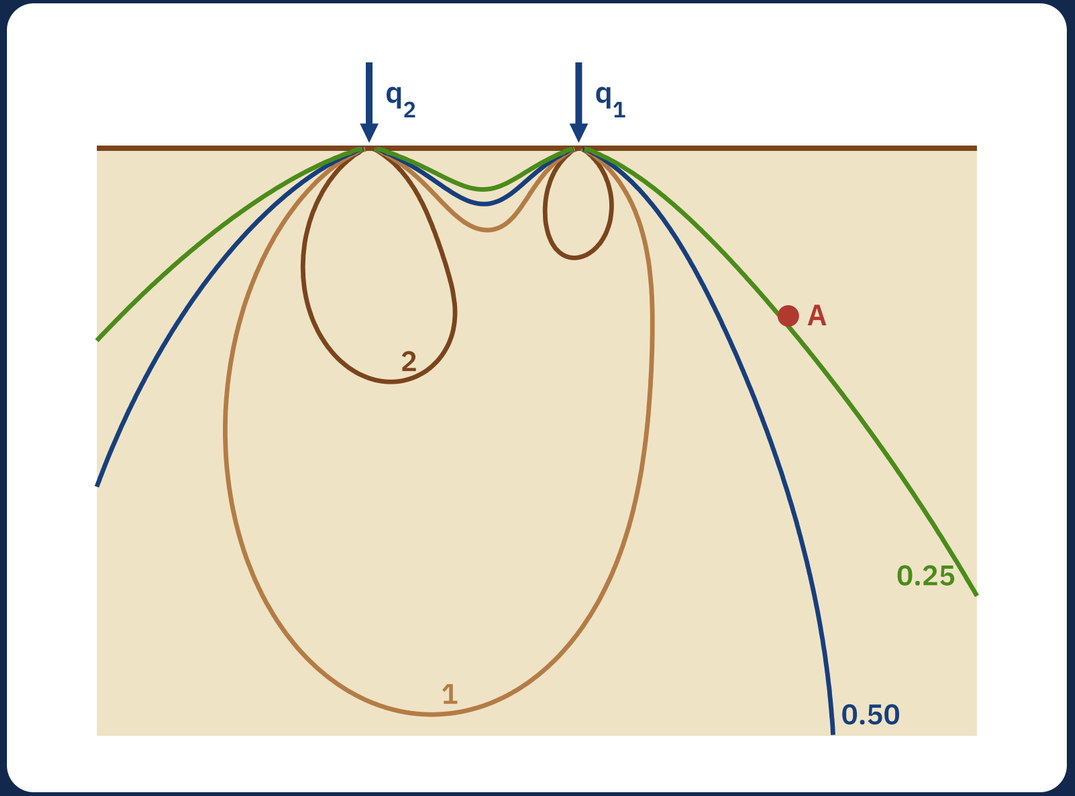
Geotechnical lecture presentation · Stress distribution in soil

Stress under a vertical line load

Lecture: vertical stress induced in the soil by a vertical line load at the surface; the opening point worked by hand end to end, depth and lateral profiles with tables.

Plane strain | Dimensionless curve | Superposition | Sequence of point loads | Iso-stress contours



Introduction

What will we learn in this lecture?

A line load under **plane strain**: the stress depends only on the horizontal distance **x** directly beneath and on the depth **z**

By summing the line load as a **series of point loads**, the relation $\Delta\sigma_z = 2qz^3/[\pi(x^2 + z^2)^2]$

Difference from a point load: beneath the load axis the stress decreases only with **1/z**, $\Delta\sigma_z = 2q/(\pi z) = 0.637\ q/z$

Dimensionless form: a single $\Delta\sigma_z/(q/z)$ curve and an x/z table; **superposition** for several parallel loads

The module's opening example (two line loads) **end to end by hand**; depth and lateral variation in tables

LECTURE FLOW

Each topic is covered in **three steps**:

1 · **Theory**: formula and rationale

2 · **Problem statement**: given and required

3 · **Solution**: step by step, with numbers

Example: q_1 = 7.5 kN/m (x = 0) and q_2 = 15 kN/m (x = −5 m); point A: x = 5 m, z = 4 m.

Load per unit length along an infinite line

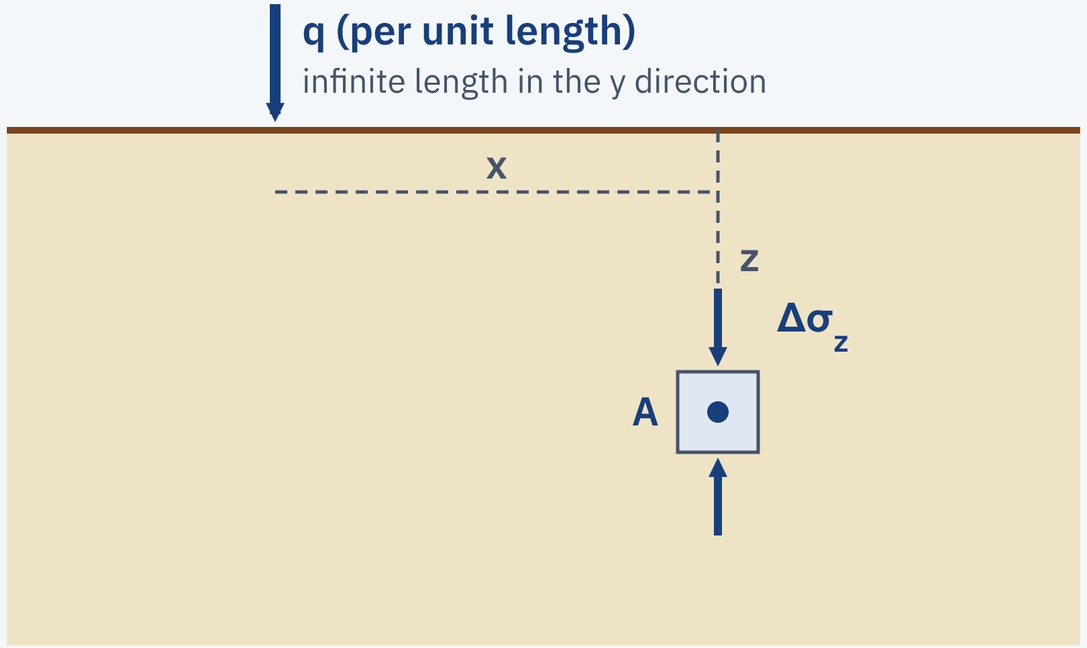
THEORY

A line load is a load of intensity q per unit length acting along an infinitely long line on the surface: a long wall, a rail, or a strip so narrow that its width can be neglected.

Since every section perpendicular to the line is loaded in the same way, the stresses are in a state of **plane strain** and depend only on the horizontal distance x and the depth z.

The load is flexible, vertical and of infinite length in the y direction; the soil is assumed to be a homogeneous, isotropic, linear elastic half-space.

$\Delta\sigma_z$ is the increase caused by the load; the overburden of the soil above the point is not included.



From a row of point loads to a line load

THEORY

A line load can be regarded as a row of point loads dP = q dy arranged along y.

Each point load gives the Boussinesq stress with $r^2 = x^2 + y^2$; when these contributions are summed along the line, the closed-form value of the integral becomes $4/[3(x^2 + z^2)^2]$.

$$\Delta\sigma_z = \int_{-\infty}^{\infty} \frac{3q\,z^3}{2\pi\,(x^2+y^2+z^2)^{5/2}}\,dy$$

$$\int_{-\infty}^{\infty} \frac{dy}{(x^2+y^2+z^2)^{5/2}} = \frac{4}{3\,(x^2+z^2)^2} \;\Rightarrow\; \Delta\sigma_z = \frac{2q\,z^3}{\pi\,(x^2+z^2)^2}$$

Unlike a point load, the stress decreases with depth only as 1/z: on the axis $\Delta\sigma_z = 2q/(\pi z) = 0.637\ q/z$.

$\Delta\sigma_z/(q/z)$ and x/z

THEORY

When divided by z and x/z is substituted, the equation takes a dimensionless form that gives a single curve for each line load.

Thus a single **$\Delta\sigma_z/(q/z)$** curve is used for every value of x/z.

$$\frac{\Delta\sigma_z}{q/z} = \frac{2}{\pi\left[(x/z)^2+1\right]^2}$$

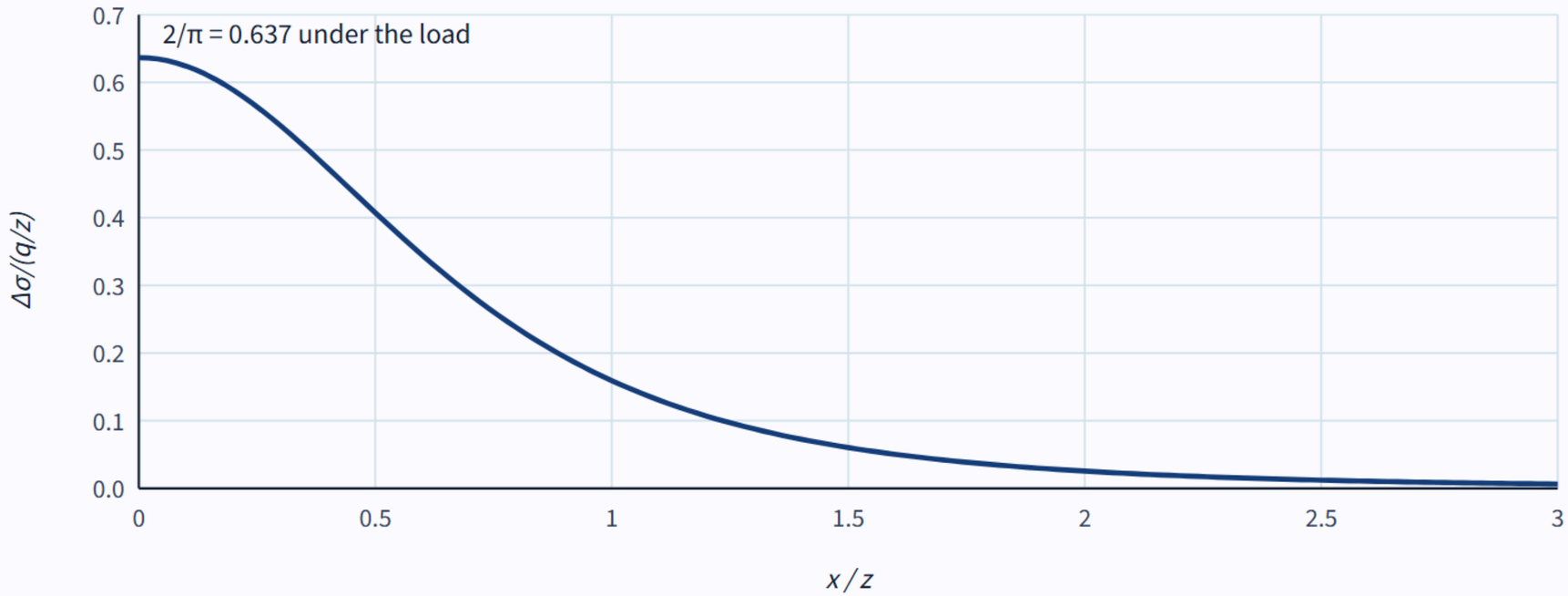
$$x = 0: \qquad \Delta\sigma_z = \frac{2q}{\pi z} = 0.637\,\frac{q}{z}$$

$\Delta\sigma_z/(q/z)$ VALUES (MODULE)

x/z	$\Delta\sigma_z/(q/z)$	x/z	$\Delta\sigma_z/(q/z)$
0.0	0.637	1.0	0.159
0.1	0.624	1.2	0.107
0.2	0.589	1.4	0.073
0.3	0.536	1.6	0.050
0.4	0.473	1.8	0.035
0.5	0.407	2.0	0.025
0.6	0.344	2.4	0.014
0.7	0.287	2.8	0.008
0.8	0.237	3.0	0.006
0.9	0.194	3.5	0.004

$\Delta\sigma_z/(q/z)$ versus x/z

THEORY



Δσz/(q/z) against x/z, from the equation. At x/z = 1 it is a quarter of its value under the load; beyond x/z = 3 it is below 0.01.

Superposition

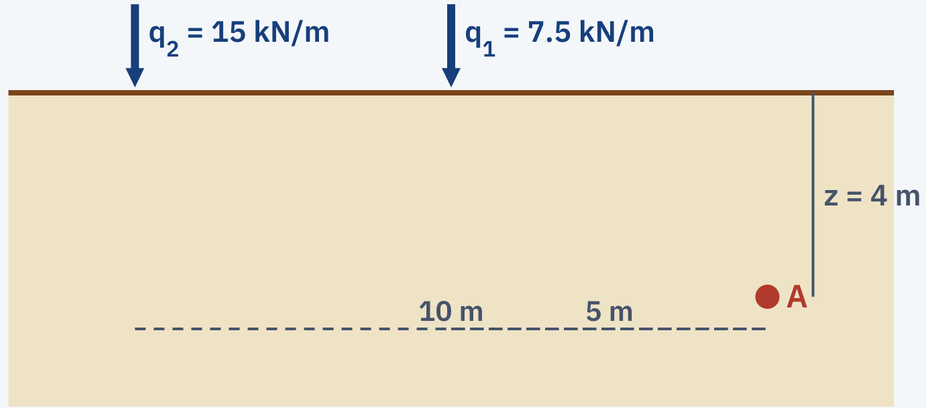
THEORY

Since the soil is elastic, the stresses of several line loads are summed by superposition.

The stress at point A is the sum of the stresses produced individually by each load, using its own horizontal distance to point A.

The module accepts at most eight parallel line loads; the loads are entered one by one with their intensities and positions via **Add line load**, and no two of them may be at the same position.

$$\Delta\sigma_z = \sum_k \frac{2\,q_k\,z^3}{\pi\,(x_k^2 + z^2)^2}$$



Problem statement

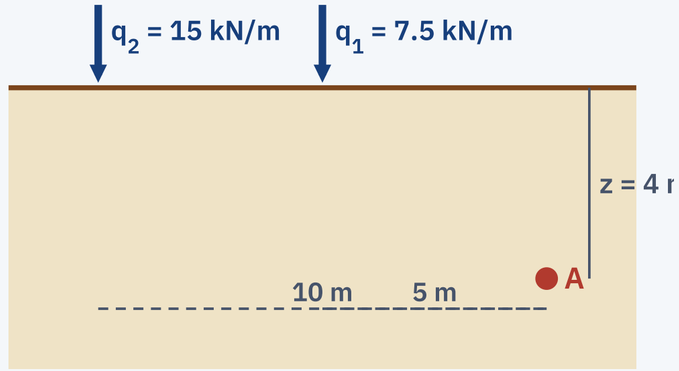
EXAMPLE

Given

Loads	$q_1 = 7.5$ **kN/m** ($x_L = 0$); $q_2 = 15$ **kN/m** ($x_L = -5$ **m**)
Point A	$x_A = 5$ **m**, $z = 4$ **m (load 1 at** 5 **m, load 2 at** 10 **m)**
Soil	**homogeneous, elastic half-space; plane strain**

REQUIRED

The **$\Delta\sigma_z$** produced by each load at point A and, by superposition, the total **$\Delta\sigma_z$**.



Solution

SOLUTION

1 Load 1: $x = x_A - x_L = 5 - 0 = 5$ m

$$\Delta\sigma_{z(1)} = \frac{2q_1 z^3}{\pi(x^2+z^2)^2} = \frac{2(7.5)(4^3)}{\pi(5^2+4^2)^2} = \frac{960}{\pi\,(1{,}681)} = \mathbf{0.18178}\ \text{kPa}$$

2 Load 2: $x = 5 - (-5) = 10$ m

$$\Delta\sigma_{z(2)} = \frac{2(15)(4^3)}{\pi(10^2+4^2)^2} = \frac{1{,}920}{\pi\,(13{,}456)} = \mathbf{0.04542}\ \text{kPa}$$

3 Superposition (soil elastic: stresses are added)

$$\Delta\sigma_z = \Delta\sigma_{z(1)} + \Delta\sigma_{z(2)} = 0.18178 + 0.04542 = \mathbf{0.22720}\ \text{kPa}$$

$\Delta\sigma_z$ = 0.22720 kPa. Although load 2 is twice as large, it is twice as far from A, so its stress is about one fifth of the total (ratio 0.20). Check for load 1: $x/z = 1.25 \Rightarrow 2/(\pi[1.25^2 + 1]^2) = 0.09695$; $q_1/z = 1.875$; $0.09695 \cdot 1.875 = 0.18178$. The module gives the same value.

Check the solution in GeoTerva ↗

Problem statement

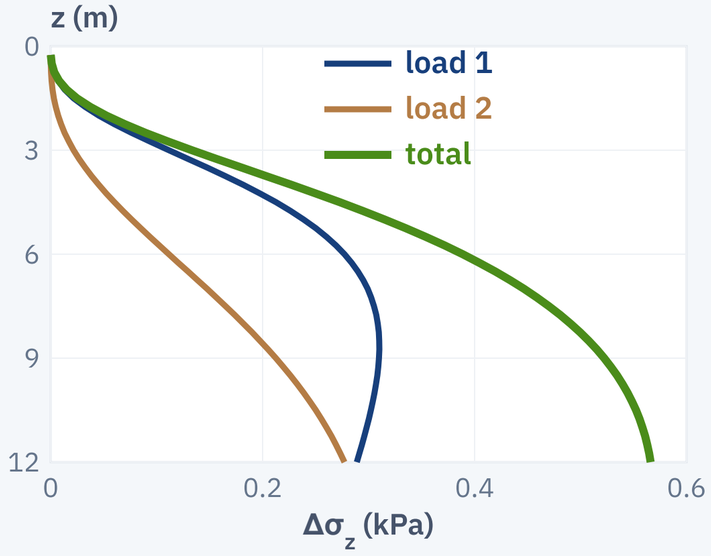
EXAMPLE

Given

Loads	**The two loads in Example 1: $q_1 = 7.5$ kN/m ($x_L = 0$), $q_2 = 15$ kN/m ($x_L = -5$ m)**
Points	**$x_A = 5$ m; $z = 1.2; 2.4;$ …; 12 m (profile rows of the module)**

REQUIRED

For each depth, **$\Delta\sigma_{z(1)}$**, **$\Delta\sigma_{z(2)}$** and the total **$\Delta\sigma_z$**; detailed solution for z = 8 m; the depth of the maximum stress.



Solution (1/2)

SOLUTION

1 Load 1 (x = 5 m): z = 8 m

$$\Delta\sigma_{z(1)} = \frac{2(7.5)(8^3)}{\pi(5^2+8^2)^2} = \frac{7{,}680}{\pi\,(7{,}921)} = \mathbf{0.30863}\ \text{kPa}$$

2 Load 2 (x = 10 m): z = 8 m

$$\Delta\sigma_{z(2)} = \frac{2(15)(8^3)}{\pi(10^2+8^2)^2} = \frac{15{,}360}{\pi\,(26{,}896)} = \mathbf{0.18178}\ \text{kPa}$$

3 Total

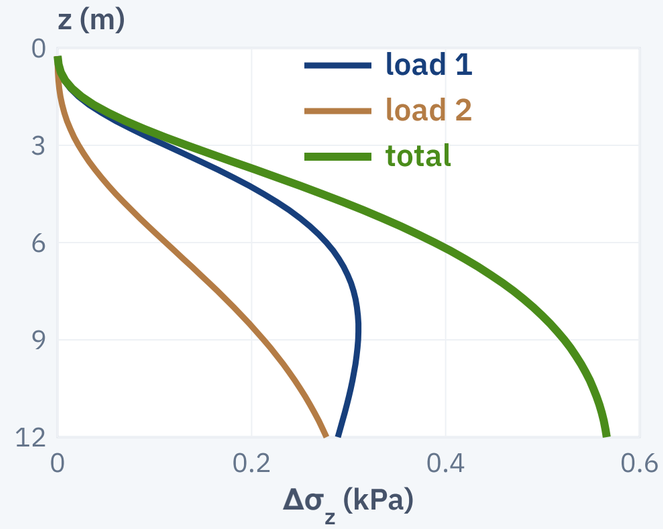
$$\Delta\sigma_z = 0.30863 + 0.18178 = \mathbf{0.49041}\ \text{kPa}$$

At z = 8 m, $\Delta\sigma_z$ = 0.49041 kPa; it is much larger than the value at z = 4 m (0.22720 kPa): point A is beside the loads and the stress continues to increase with depth (the table is on the next page).

Solution (2/2)

SOLUTION

z (m)	$\Delta\sigma_{z(1)}$ (kPa)	$\Delta\sigma_{z(2)}$ (kPa)	Total $\Delta\sigma_z$ (kPa)
1.2	0.01180	0.00160	**0.01341**
2.4	0.06976	0.01180	**0.08156**
3.6	0.15460	0.03492	**0.18951**
4.8	0.22880	0.06976	**0.29856**
6.0	0.27716	0.11152	**0.38868**
7.2	0.30183	0.15460	**0.45643**
8.4	0.30990	0.19456	**0.50446**
9.6	0.30775	0.22880	**0.53655**
10.8	0.29981	0.25631	**0.55612**
12.0	0.28888	0.27716	**0.56604**



For a single load the stress is greatest at depth $z = \sqrt{3}\cdot x$: for load 1, at 8.66 m (0.3101 kPa). The sum of the two loads is still increasing at 12 m; the maximum total is 0.5687 kPa at a depth of 13.1 m.

Check the solution in GeoTerva ↗

Problem statement

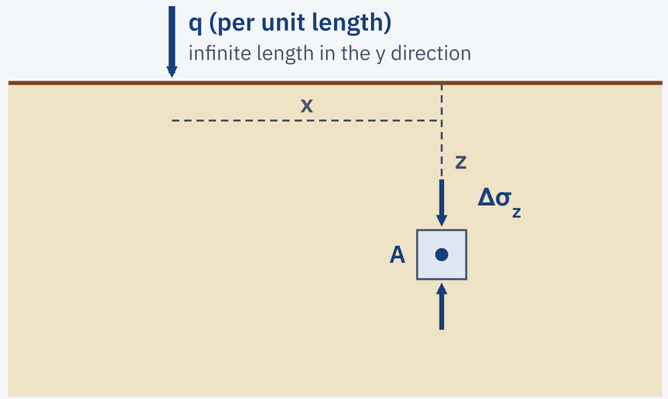
EXAMPLE

Given

Load	$q = 20$ **kN/m, vertical line load**
Points	$z = 3$ **m;** $x = 0; 1.5; 3; 4.5; 6$ **m**

REQUIRED

For each point, **x/z**, **$\Delta\sigma_z/(q/z)$** and **$\Delta\sigma_z$**; the ratio of the value at point x = z to the value under the load.



Solution (1/2)

SOLUTION

1 Under the load (x = 0): x/z = 0

$$\Delta\sigma_z = \frac{2q}{\pi z} = \frac{2(20)}{\pi(3)} = \mathbf{4.2441}\ \text{kPa}$$

2 x = 3 m: x/z = 1

$$\frac{\Delta\sigma_z}{q/z} = \frac{2}{\pi\left[1^2+1\right]^2} = \frac{2}{4\pi} = 0.15915$$

$$\Delta\sigma_z = 0.15915 \cdot \frac{20}{3} = 6.6667 \cdot 0.15915 = \mathbf{1.0610}\ \text{kPa}$$

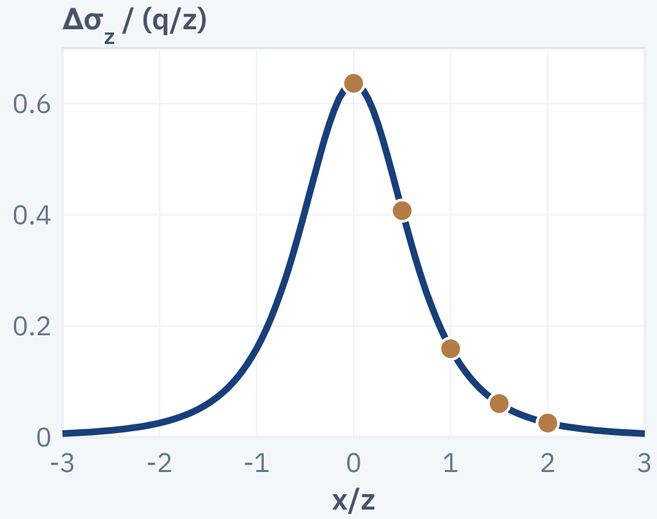
At the point x/z = 1 the stress is **one quarter** of that under the load (1.0610 / 4.2441 = 0.25).

Solution (2/2)

SOLUTION

x (m)	x/z	$\Delta\sigma_z/(q/z)$	$\Delta\sigma_z$ (kPa)
0.0	0.0	0.63662	4.2441
1.5	0.5	0.40744	2.7162
3.0	1.0	0.15915	**1.0610**
4.5	1.5	0.06027	0.4018
6.0	2.0	0.02546	0.1698

$\Delta\sigma_z/(q/z)$ depends only on x/z: the same curve is used for every q and every z. At x/z = 2 the stress is 0.04 times that beneath the load (0.1698 kPa).



Check the solution in GeoTerva ↗

Problem statement

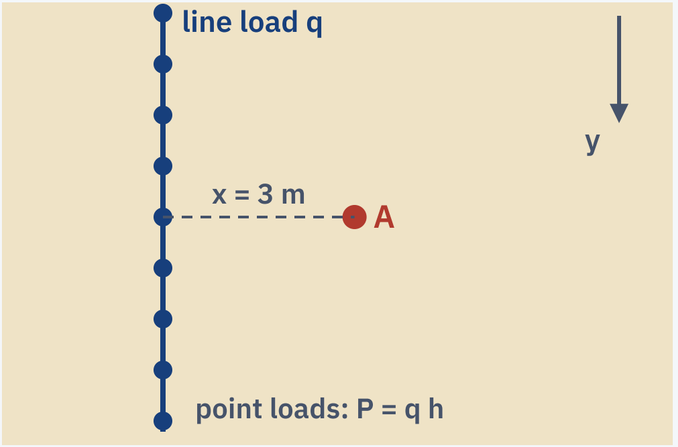
EXAMPLE

Given

Load	$q = 10$ **kN/m; discrete loads** $P = q\,h$ **spaced** h **along the line (**$h = 8; 4; 2; 1$ **m)**
Point A	$x = 3$ **m,** $z = 4$ **m (horizontal distance of the line); Boussinesq:** $\Delta\sigma_z = 3Pz^3/(2\pi L^5)$

REQUIRED

Sum of the Boussinesq stresses of the point loads (term by term for h = 2 m); approach to the line load value as h decreases.



Solution (1/2)

SOLUTION

1 $P = q\,h = 10(2) = 20$ kN; for each load $L_k^2 = x^2 + y_k^2 + z^2 = 9 + y_k^2 + 16$

k	y_k (m)	$L_k^2 = 25 + y_k^2$	L_k (m)	$\Delta\sigma_{z,k}$ (kPa)	How many loads
0	0	25	5.000	0.19557	1
1	2	29	5.385	0.13494	2 (±k)
2	4	41	6.403	0.05678	2 (±k)
3	6	61	7.810	0.02103	2 (±k)
4	8	89	9.434	0.00818	2 (±k)
5	10	125	11.180	0.00350	2 (±k)

Example terms: k = 0: $\Delta\sigma_{z,0} = 3(20)(4^3)/[2\pi(5)^5] = 0.19557$ kPa; the sum for $|k| \le 5$ is 0.64443 kPa, and the remainder for $|k| \ge 6$ is 0.00748 kPa.

Solution (2/2)

SOLUTION

h (m)	P = q h (kN)	Sum of the individual loads $\Delta\sigma_z$ (kPa)	Difference (%)
8	80	0.852175	30.72
4	40	0.660644	1.34
2	20	0.651907	0.0013
1	10	0.651899	< 0.0001
line load		**0.651899**	0

For a line load, $\Delta\sigma_z = 2q\,z^3/[\pi(x^2 + z^2)^2] = 2(10)(4^3)/[\pi(25)^2]$ = **0.651899 kPa**. When point loads are taken closely spaced as $h \le z/2$, the total approaches this value closely (h = 2 m: 0.651907 kPa); for a sparse spacing such as h = 8 m, however, the difference is large. This shows that the line load formula is the integral of a row of point loads.

Check the solution in GeoTerva ↗

Problem statement

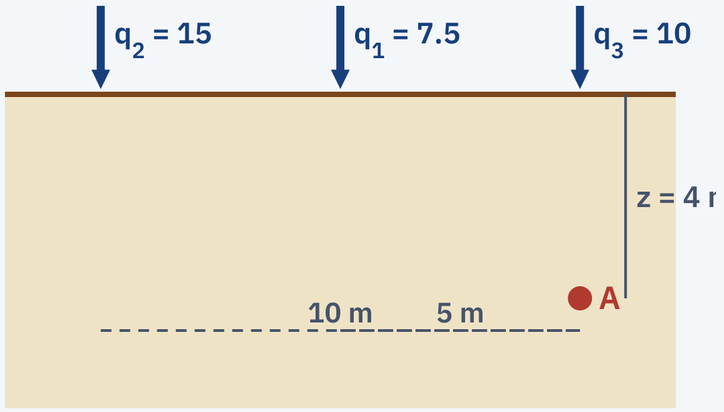
EXAMPLE

Given

Loads	$q_1 = 7.5$ kN/m ($x_L = 0$); $q_2 = 15$ kN/m ($x_L = -5$ m); $q_3 = 10$ kN/m ($x_L = 5$ m)
Points	A: $x_A = 5$ m, $z = 4$ m (directly below load 3); at $z = 4$ m, $x = -7.5$ … 7.5 m

REQUIRED

The contribution of each load at point A and the total $\Delta\sigma_z$; the total stress along x at depth z = 4 m.



Solution (1/2)

SOLUTION

1. Load 3 (x = 5 – 5 = 0): A is directly below the load

$$\Delta\sigma_{z(3)} = \frac{2q_3}{\pi z} = \frac{2(10)}{\pi(4)} = \mathbf{1.59155}\ \mathrm{kPa}$$

2. Load 1 (x = 5 m) and load 2 (x = 10 m): values from Example 1

$$\Delta\sigma_{z(1)} = 0.18178 \qquad \Delta\sigma_{z(2)} = 0.04542\ \mathrm{kPa}$$

3. Superposition

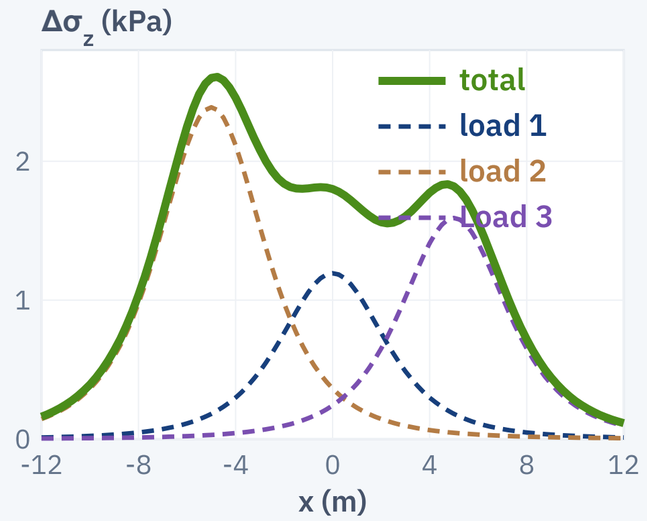
$$\Delta\sigma_z = 1.59155 + 0.18178 + 0.04542 = \mathbf{1.81875}\ \mathrm{kPa}$$

Load 3 contributes **87.5 percent** of the total: beneath a line load its own stress dominates, while distant loads make a small contribution. The third load adds 1.592 kPa to the 0.2272 kPa value of Example 1.

Solution (2/2)

SOLUTION

x (m)	Load 1	Load 2	Load 3	Total $\Delta\sigma_z$ (kPa)
−7.5	0.0585	1.2345	0.0137	1.3068
−5.0	0.1818	2.3873	0.0303	2.5994
−2.5	0.6173	1.2345	0.0781	1.9298
0.0	1.1937	0.3636	0.2424	1.7996
2.5	0.6173	0.1171	0.8230	1.5573
5.0	0.1818	0.0454	1.5915	**1.8188**
7.5	0.0585	0.0206	0.8230	0.9021



The total curve gives a peak near each load (x ≈ −4.75; −0.50; 4.75 m); the peaks shift slightly due to the tails of the adjacent loads. The highest peak is around the most severe load (load 2, 15 kN/m): **2.5994 kPa** at x = −5 m; at point A (x = 5 m) 1.8188 kPa.

Check the solution in GeoTerva ↗

Problem statement

EXAMPLE

Given

Loads	**line load $q = 50$ kN/m; point load $P = 50$ kN**
Points	**under the load ($x = 0$, $r = 0$); $z = 1$; 2; 3; 5; 10 m**

REQUIRED

$\Delta\sigma_z$ beneath the axis and the rate of decrease with depth for the two load types.

Solution

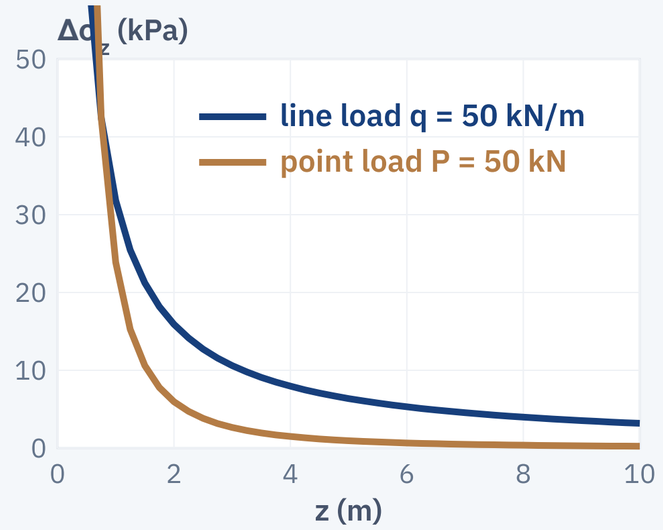
SOLUTION

1 z = 5 m

$$\text{line: } \Delta\sigma_z = \frac{2q}{\pi z} = \frac{2(50)}{\pi(5)} = \mathbf{6.366}\ \text{kPa}$$

$$\text{single: } \Delta\sigma_z = \frac{3}{2\pi}\frac{P}{z^2} = 0.4775\,\frac{50}{5^2} = \mathbf{0.955}\ \text{kPa}$$

z (m)	Line load: 2q/(πz) (kPa)	Point load: 0.4775 P/z² (kPa)	Ratio
1	31.831	23.873	1.3
2	15.915	5.968	2.7
3	10.610	2.653	4.0
5	6.366	0.955	6.7
10	3.183	0.239	13.3



As z goes from 1 → 10 m, the line-load stress decreases by a factor of **10** (31.83 → 3.18 kPa) and the point-load stress by a factor of **100** (23.87 → 0.24 kPa).

Check the solution in GeoTerva ↗

Summary

Key points

Line load (plane strain): $\Delta\sigma_z = 2q\,z^3/[\pi(x^2 + z^2)^2] = (q/z) \cdot 2/\{\pi[(x/z)^2 + 1]^2\}$; under the load $\Delta\sigma_z = 2q/(\pi z) = 0.637\ q/z$

Row of point loads: the Boussinesq stresses of the point loads $dP = q\,dy$ are summed; point loads spaced $h = 2$ m give 0.65191 kPa, the line load 0.65190 kPa

Example ($q_1 = 7.5$ kN/m, $q_2 = 15$ kN/m; A: $x = 5$, $z = 4$ m): $\Delta\sigma_{z(1)} = 0.18178$; $\Delta\sigma_{z(2)} = 0.04542$; **$\Delta\sigma_z = 0.22720$ kPa**

Dimensionless curve: for $x/z = 1$ the value is one quarter of that under the load; **superposition**: the stress from each load is found with its own horizontal distance and summed (for three loads, 1.8188 kPa at A)

Depth: under a load the stress decreases with $1/z$ ($1/z^2$ for a point load); at a point beside the load the stress increases down to a depth of $z = \sqrt{3}\cdot x$

References

Das, B.M. (2018). Principles of Geotechnical Engineering, 9th ed. Cengage Learning.

Boussinesq, J. (1883). Application des Potentiels... Gauthier-Villars, Paris.

NOTE

All the numerical examples have been produced with the module's own calculation engine; the module gives the same values. The module opens with two line loads (7.5 and 15 kN/m) and point A (x = 5 m, z = 4 m).

GeoTerva

Online geotechnical analysis

From theory to practice

Enter your own line loads, their positions and your point in GeoTerva; the module gives the contribution of each load, the total $\Delta\sigma_z$ value, the depth profile, isostress contours, the step-by-step solution and the report together.

Open the module → **Theory page →**

Open on your phone
The example modules need no registration.

If you would like to use them in your lectures: info@geoterva.com