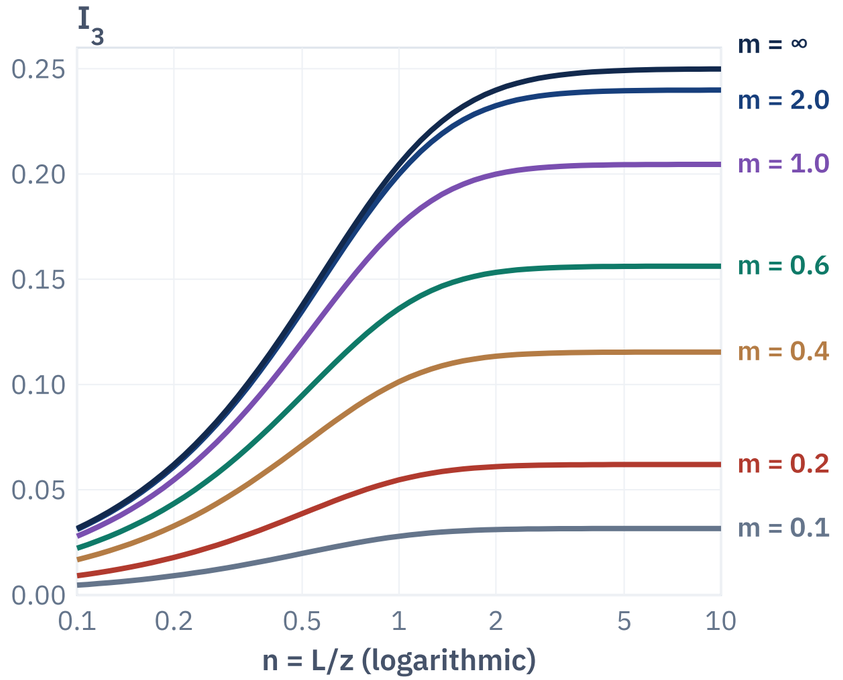
GeoTerva

Geotechnical lecture presentation · Stresses in the soil mass

Stress under a rectangular load

Lecture: vertical stress under a rectangular load; corner solution, superposition and hand-worked examples.

- Under the corner · I_3
- Four rectangles
- Under the center · I_4
- Outside the point load
- Multiple rectangles



What will we learn in this lecture?

The vertical stress increase beneath a **rectangular area** with a uniformly distributed load (footing, raft, slab): $\Delta\sigma_z$

Closed-form solution under the corner: $\Delta\sigma_z = q\, I_3$, I_3 depends only on $m = B/z$ and $n = L/z$

For **any point**, the **addition and subtraction** of rectangles: the load is divided into four rectangles having a common corner above the point

Directly below the center, I_4; for a square, check $I_4 = 4\, I_3(1, 1) = 0.701$

That the stresses from multiple rectangles are **superposed**; the effect of the rectangle shape (L/B)

LECTURE FLOW

Each topic is covered in **three steps**:

1 · **Theory**: formula and rationale

2 · **Problem statement**: given and required

3 · **Solution**: step by step, with numbers

Example: 3 m × 2 m rectangle, q = 150 kN/m²; point A′ is 1 m beyond the short side, along one of the long sides; z = 4 m.

Rectangular area with uniformly distributed load

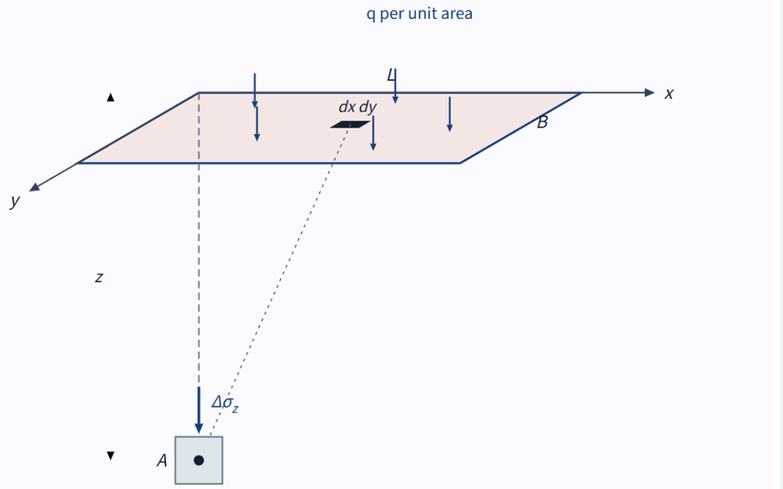
THEORY

Most footings and rafts are **rectangular**; when the Boussinesq solution is integrated over a rectangle, the vertical stress **below a corner** is obtained in closed form

For all other points the rectangles are **added and subtracted**

A flexible rectangle of dimensions **L × B** carrying a uniformly distributed pressure **q**; the soil is a homogeneous, isotropic, linearly elastic semi-infinite medium

$\Delta\sigma_z$ is the stress increase due to the load alone



Boussinesq integral: $\Delta\sigma_z = q\, I_3$

THEORY

The elementary area carries the load **dq = q dx dy**; in the Boussinesq equation P = q dx dy and r² = x² + y² are taken

$$d\sigma_z = \frac{3q\,dx\,dy\,z^3}{2\pi\left(x^2+y^2+z^2\right)^{5/2}}$$

$$\Delta\sigma_z = \int_0^B\int_0^L \frac{3q\,z^3\,dx\,dy}{2\pi\left(x^2+y^2+z^2\right)^{5/2}} = q\,I_3$$

m = B/z and **n = L/z**; I_3 depends only on these two ratios

Influence factor I_3

THEORY

$$I_3 = \frac{1}{4\pi}\left[\frac{2mn\sqrt{m^2+n^2+1}}{m^2+n^2+m^2n^2+1}\cdot\frac{m^2+n^2+2}{m^2+n^2+1} + \tan^{-1}\left(\frac{2mn\sqrt{m^2+n^2+1}}{m^2+n^2-m^2n^2+1}\right)\right]$$

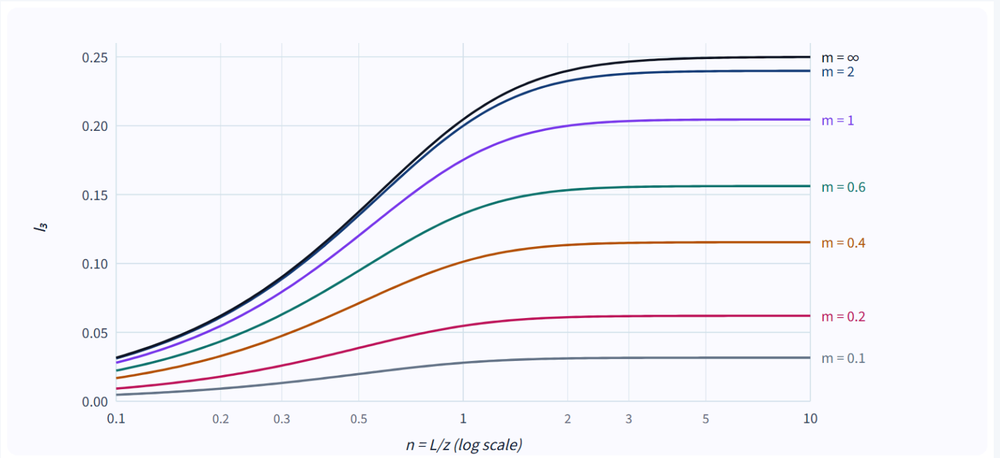
The arctangent must be a **positive** angle in radians: when $m^2 + n^2 + 1 < m^2n^2$ it comes out negative and **π is added**

It is **symmetric** with respect to I_3, m and n; for a very large rectangle it approaches **1/4**: one corner carries one quarter of the load of an infinite area

I_3 values

THEORY

n \ m	0.1	0.2	0.5	1.0	2.0	3.0
0.10	0.0047	0.0092	0.0198	0.0279	0.0311	0.0315
0.25	0.0113	0.0220	0.0475	0.0674	0.0752	0.0761
0.50	0.0198	0.0387	0.0840	0.1202	0.1350	0.1368
1.00	0.0279	0.0547	0.1202	0.1752	0.1999	0.2034
2.00	0.0311	0.0610	0.1350	0.1999	0.2325	0.2378
5.00	0.0316	0.0620	0.1374	0.2044	0.2395	0.2461



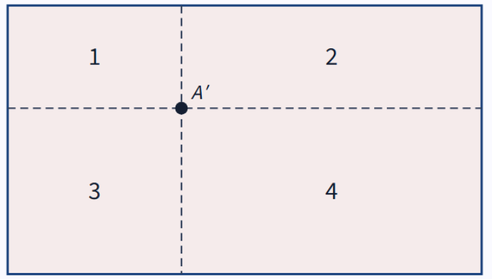
Four-corner rectangle

THEORY

$$\Delta\sigma_z = q\left[I_{3(1)} + I_{3(2)} + I_{3(3)} + I_{3(4)}\right]$$

The loaded area is divided into rectangles having a **common corner** above point A′; the corner stresses are summed

If A′ is **outside** the area, part of the rectangles extends beyond the area and is **removed**; the module draws the four rectangles from the point load to the corners with signed distances at each point, and each I_3 takes the sign of the product of its two sides



plan of the loaded area; A′ is a corner of all four rectangles 1–4

Below the center: I_4

THEORY

$$\Delta\sigma_z = q\, I_4, \quad I_4 = \frac{2}{\pi}\left[\frac{m_1 n_1}{\sqrt{1+m_1^2+n_1^2}}\,\frac{1+m_1^2+2n_1^2}{(1+n_1^2)(m_1^2+n_1^2)} + \sin^{-1}\frac{m_1}{\sqrt{m_1^2+n_1^2}\,\sqrt{1+n_1^2}}\right]$$

Below the center the four rectangles are **equal**; the result is written directly: **m_1 = L/B**, **n_1 = z/b**, b = B/2 (L ≥ B)

For a square at depth z = b, **I_4 = 0.701 = 4 I_3(1, 1)**

Superposition

THEORY

Each rectangle is defined by its own **q, L, B** and its center on the ground (x_c, y_c) (at most eight in the module)

Two rectangles **may share an edge but cannot overlap**

Since the soil is elastic, the stress of each rectangle is found from its own **four corner rectangles** and the stresses are summed

$$\Delta\sigma_z = \sum_k q_k \sum \pm I_3 \text{ (four corners)}$$

Problem statement

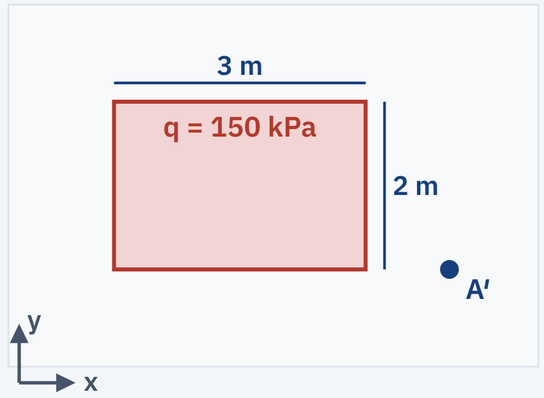
EXAMPLE

Given

Rectangular	3×2 m ($L = 3$ m in the x direction, $B = 2$ m in the y direction), $q = 150$ kN/m², centered at the origin
Point A′	1 m beyond the short side, in the direction of one of the long sides: $x = 2.5$ m, $y = -1$ m; $z = 4$ m

REQUIRED

$\Delta\sigma_z$: by the four-corner rectangle; compute the two I_3 values in detail.



Solution (1/3)

SOLUTION

1 Marked distances from A′ to the four corners of the load (z = 4 m)

Corner (x; y)	a = x − $x_{A'}$	b = y − $y_{A'}$	m = |a|/z	n = |b|/z	sign
(1.5; 1.0)	−1.0	2.0	0.25	0.50	–
(1.5; -1.0)	−1.0	0.0	0.25	0.00	0 (b = 0)
(-1.5; 1.0)	−4.0	2.0	1.00	0.50	+
(-1.5; -1.0)	−4.0	0.0	1.00	0.00	0 (b = 0)

2 Two rectangles remain: A′ common corner

$$\Delta\sigma_z = q\,[I_3(0.5;\,1) - I_3(0.25;\,0.5)] \qquad 4\ \mathrm{m} \times 2\ \mathrm{m} \;-\; 1\ \mathrm{m} \times 2\ \mathrm{m}$$

Two corners with b = 0 do not contribute. For the **4 m × 2 m** rectangle, m = B/z = 2/4 = 0.5; n = L/z = 4/4 = 1; for the **1 m × 2 m** rectangle, m = 1/4 = 0.25; n = 2/4 = 0.5. Since I_3 is symmetric, (1; 0.5) and (0.5; 1) are the same.

Solution (2/3)

SOLUTION

Interpolation	$I_3(0.5; 1)$	$I_3(0.25; 0.5)$
m ; n	0.50 ; 1.00	0.25 ; 0.50
m^2 ; n^2	0.2500 ; 1.0000	0.0625 ; 0.2500
$S_1 = m^2 + n^2 + 1$	2.2500	1.3125
$\sqrt{S_1}$	1.5000	1.1456
$2mn\sqrt{S_1}$	1.5000	0.2864
$S_2 = m^2 + n^2 + m^2n^2 + 1$	2.5000	1.3281
$(m^2 + n^2 + 2)/S_1$	1.4444	1.7619
1st term = $2mn\sqrt{S_1} / S_2 \cdot (m^2 + n^2 + 2)/S_1$	0.8667	0.3800
$D = m^2 + n^2 + 1 - m^2n^2$	2.0000	1.2969
$2mn\sqrt{S_1} / D$	0.7500	0.2208
$\tan^{-1}(\ldots)$ (rad)	0.6435	0.2174
I_3 = (1st term + $\tan^{-1}$)/(4π)	**0.12018**	**0.04753**

$I_3(0.5; 1) = 0.12018$ and **$I_3(0.25; 0.5) = 0.04753$**. Since D > 0 at both angles, the arctangent is positive; there is no need to add π.

Solution (3/3)

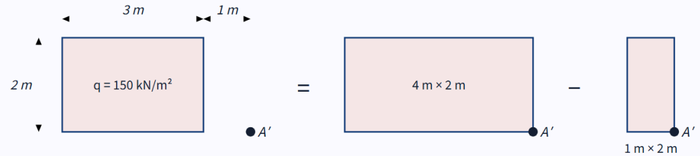
SOLUTION

3 Difference and stress

$$\Delta\sigma_z = 150\,(0.12018 - 0.04753) = 150\,(0.07264)$$

$$\Delta\sigma_z = 18.03 - 7.13 = \mathbf{10.90}\ \mathrm{kPa}$$

At point A′, **$\Delta\sigma_z$ = 10.90 kPa** ($\Delta\sigma_z/q$ = 0.07264). The module opens with this example; if the rectangle and the point are shifted together by the same distance (center 4; 3 m, point 6.5; 2 m), the stress does not change: 10.90 kPa.



Check the solution in GeoTerva ↗

Problem statement

EXAMPLE

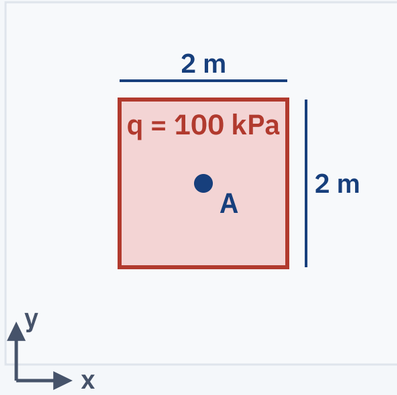
Given

(a) Square — 2×2 m, $q = 100$ kN/m²; below the center $z = 1$ m ($n_1 = z/b = 1$)

(b) Rectangular — 3×2 m, $q = 150$ kN/m²; below the center $z = 4$ m ($m_1 = L/B = 1.5$; $n_1 = 4$)

REQUIRED

(a) check $\mathbf{I_4 = 4\, I_3(1, 1)}$; (b) $\mathbf{I_4}$ and $\mathbf{\Delta\sigma_z}$; comparison with the sum of the four corners.



Solution (1/2)

SOLUTION

1 Four equal rectangles (each 1 m × 1 m, z = 1 m): $I_3(1, 1)$; $S_1 = 3$, $\sqrt{S_1} = 1.7321$, $2mn\sqrt{S_1} = 3.4641$, $S_2 = 4$

$$\text{1st term} = \frac{3.4641}{4} \cdot \frac{4}{3} = 1.1547 \qquad \tan^{-1}\frac{3.4641}{3-1} = \tan^{-1}(1.7321) = 1.0472$$

$$I_3 = \frac{1.1547 + 1.0472}{4\pi} = \mathbf{0.17522}$$

2 Check with the centre formula ($m_1 = 1$; $n_1 = 1$)

$$4\,I_3 = 4\,(0.17522) = 0.70089$$

$$I_4 = \frac{2}{\pi}\left[\frac{1}{\sqrt{3}} \cdot \frac{4}{4} + \sin^{-1}\frac{1}{2}\right] = \frac{2}{\pi}\,(0.5774 + 0.5236) = \mathbf{0.70089}$$

For a square, **$I_4 = 4\ I_3(1, 1) = 0.70089 \approx 0.701$**. In this example, the stress below the center equals 70.1% of the load pressure (z = b).

Solution (2/2)

SOLUTION

1 I_4: $m_1 = L/B = 1.5$; $n_1 = z/b = 4$ ($b = 1$ m); $m_1^2 + n_1^2 = 18.25$

$$\frac{m_1 n_1}{\sqrt{1 + m_1^2 + n_1^2}} = \frac{6}{4.3875} = 1.3675 \qquad \frac{1 + m_1^2 + 2n_1^2}{(1 + n_1^2)(m_1^2 + n_1^2)} = \frac{35.25}{310.25} = 0.1136$$

$$\sin^{-1} \frac{1.5}{\sqrt{18.25}\sqrt{17}} = 0.0853 \qquad I_4 = \frac{2}{\pi}(0.1554 + 0.0853) = \mathbf{0.15320}$$

2 Check with $\Delta\sigma_z$ and the sum of the four corners

$$\Delta\sigma_z = 150\,(0.15320) = \mathbf{22.98}\ \text{kPa} \qquad 4\,I_3(0.25;\ 0.375) = 4\,(0.03830) = 0.15320$$

Under the center **$\Delta\sigma_z$ = 22.98 kPa** ($\Delta\sigma_z/q = 0.15320$). The same result is also given by four equal corner rectangles (each 1.5 m × 1 m): 4 I_3(0.25; 0.375).

Check the solution in GeoTerva ↗

Problem statement

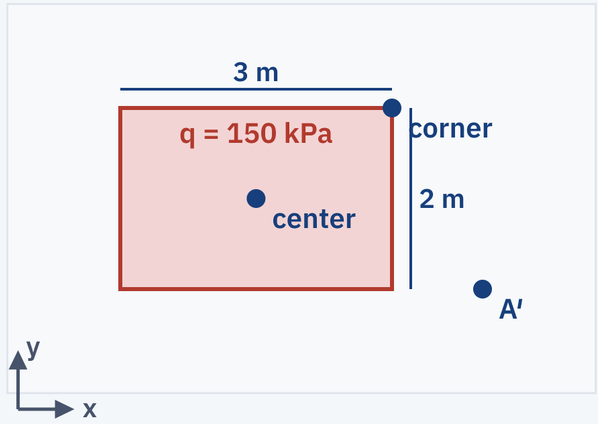
EXAMPLE

Given

Rectangular	3×2 **m,** $q = 150$ **kN/m²**
Points	**centre** $(0; 0)$**, corner** $(1.5;\ 1)$ **and A′** $(2.5;\ -1)$
Depths	$z = 1; 2; 4; 6; 8; 12$ **m**

REQUIRED

$\Delta\sigma_z$ and $\Delta\sigma_z/q$ at three points; interpretation of the variation with depth.



Solution (1/2)

SOLUTION

z (m)	Center $\Delta\sigma_z$	Corner $\Delta\sigma_z$	A′ $\Delta\sigma_z$	Center /q	Corner /q	A′ /q
1	116.19	35.67	5.88	0.7746	0.2378	0.0392
2	64.24	29.05	11.96	0.4283	0.1936	0.0798
4	22.98	16.06	10.90	0.1532	0.1071	0.0726
6	11.10	9.19	7.41	0.0740	0.0612	0.0494
8	6.44	5.74	5.03	0.0429	0.0383	0.0335
12	2.93	2.78	2.60	0.0195	0.0185	0.0174

Under the **center** and **corner**, the stress decreases with depth in one direction (22.98 and 16.06 kPa at z = 4 m); the corner/center ratio is 0.70 at z = 4 m and 0.95 at z = 12 m. At **A′** (outside the load), it is 5.88 kPa at z = 1 m, rises to its maximum value (12.64 kPa) at z ≈ 2.50 m, then decreases.

Solution (2/2)

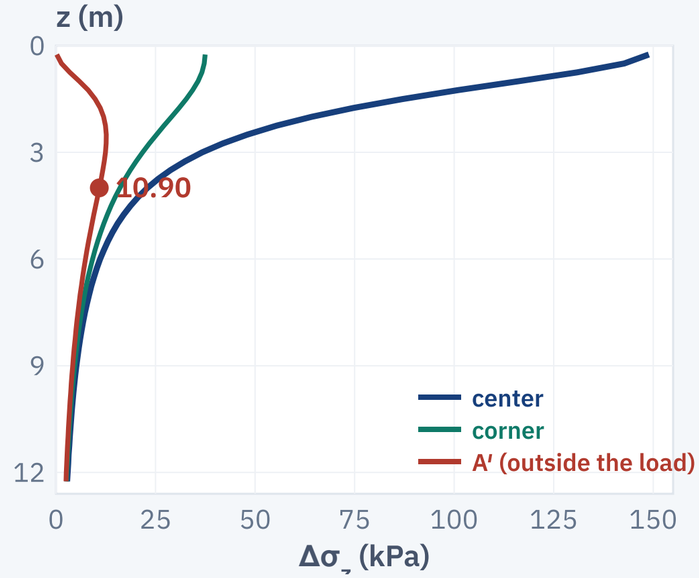
SOLUTION

1 z = 4 m row: three points

$$\text{center:}\ \ 150\,(0.15320) = 22.98\ \text{kPa}$$

$$\text{corner:}\ \ 150\,I_3(0.5;\,0.75) = 16.06\ \text{kPa} \qquad \text{A}' :\ \ 10.90\ \text{kPa}$$

Near the surface, below the center it approaches ≈ q and below the corner ≈ q/4; at depth the three curves approach each other: at z = 12 m, 2.93, 2.78 and 2.60 kPa.



Check the solution in GeoTerva ↗

Problem statement

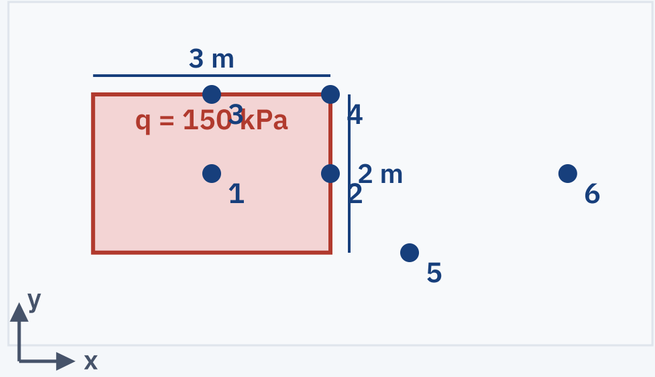
EXAMPLE

Given

Rectangular	3×2 **m,** $q = 150$ **kN/m²,** $z = 4$ **m**
Points	**(1) center; (2) midpoint of the short side; (3) midpoint of the long side; (4) corner; (5) A′; (6)** $(4.5;\ 0)$

REQUIRED

Signed I_3 terms at each point, the total and **$\Delta\sigma_z$**.



Solution (1/2)

SOLUTION

Point	(x; y)	Term 1	Term 2	Term 3	Term 4	$\Sigma \pm I_3$	$\Delta\sigma_z$ (kPa)
1	(0.0; 0)	0.0383	0.0383	0.0383	0.0383	0.1532	**22.98**
2	(1.5; 0)	0	0	0.0603	0.0603	0.1205	**18.08**
3	(0.0; 1)	0	0.0675	0	0.0675	0.1351	**20.26**
4	(1.5; 1)	0	0	0	0.1071	0.1071	**16.06**
5	(2.5; -1)	−0.0475	0	0.1202	0	0.0726	**10.90**
6	(4.5; 0)	−0.0603	−0.0603	0.0733	0.0733	0.0261	**3.91**

At the interior point (1) all four terms are **+**; at the edge points (2, 3) two terms are zero; at the corner (4) a single term remains; outside (5, 6) some terms become **−**. Ordering: 22.98 > 18.08 > 20.26 > 16.06 > 10.90 > 3.91 kPa.

Solution (2/2)

SOLUTION

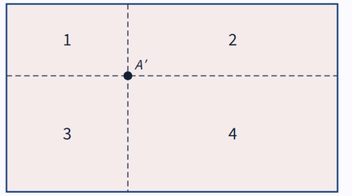
1 Corner (4): a single rectangle (3 m × 2 m)

$$\Delta\sigma_z = q\, I_3(m = 0.5;\ n = 0.75) = 150\,(0.10707) = \mathbf{16.06}\ \text{kPa}$$

2 Far point (6): (4.5; 0), 3 m beyond the load

$$\Delta\sigma_z = q \cdot 2\,[\,I_3(1.5;\ 0.25) - I_3(0.75;\ 0.25)\,]$$
$$= 150 \cdot 2\,(0.07329 - 0.06026) = \mathbf{3.91}\ \text{kPa}$$

Corner / center stress ratio 0.699; at 3 m from the load (6) 3.91 kPa remains: stress is not zero outside the loaded area either.



plan of the loaded area; A′ is a corner of all four rectangles 1–4

Check the solution in GeoTerva ↗

Problem statement

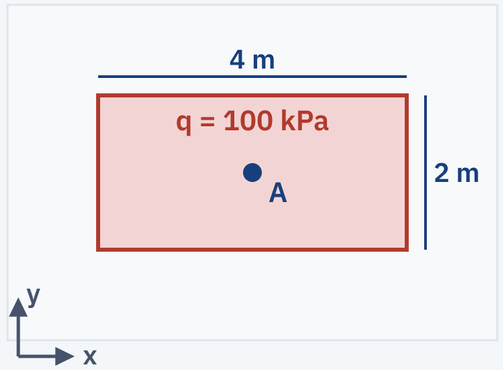
EXAMPLE

Given

Rectangular	$B = 2$ **m** ($b = 1$ **m**), $L/B = 1; 2; 5; 10$ **and** $L \to \infty$ **(strip)**
Points	**below the center,** $n_1 = z/b = 0.5; 1; 2; 4$

REQUIRED

Below the center, $\Delta\sigma_z/q = I_4$; comparison with the strip.



Solution (1/2)

SOLUTION

$n_1 = z/b$	L/B = 1	L/B = 2	L/B = 5	L/B = 10	strip (L/B → ∞)
0.5	0.9299	0.9565	0.9594	0.9595	**0.9595**
1.0	0.7009	0.7998	0.8176	0.8183	**0.8183**
2.0	0.3361	0.4807	0.5451	0.5495	**0.5498**
4.0	0.1081	0.1901	0.2848	0.3034	**0.3058**

Near the surface ($n_1 = 0.5$) the shape matters little (0.930 – 0.959); at depth the stress under a **short** rectangle decays much more rapidly: at $n_1 = 4$, the square gives 0.108, the strip 0.306.

Solution (2/2)

SOLUTION

1 I_4 formula (m_1 = 2; n_1 = 1)

$$m_1^2 + n_1^2 = 5 \qquad \frac{m_1 n_1}{\sqrt{1+5}} = \frac{2}{2.4495} = 0.8165 \qquad \frac{1+4+2}{(1+1)(5)} = 0.70$$

$$\sin^{-1}\frac{2}{\sqrt{5}\sqrt{2}} = 0.6847 \qquad I_4 = \frac{2}{\pi}(0.5715 + 0.6847) = \mathbf{0.79976}$$

2 With four equal corner rectangles (each 2 m × 1 m, z = 1 m)

$$4\,I_3(1;\,2) = 4\,(0.19994) = \mathbf{0.79976}$$

At the center of an L/B = 2 rectangle, at depth z = b, **$\Delta\sigma_z/q$ = 0.7998** (square: 0.7009; strip: 0.8183). For L/B = 10 (n_1 = 1) the value is 0.81826, strip 0.81831: as the rectangle lengthens it approaches the strip.

Check the solution in GeoTerva ↗

Problem statement

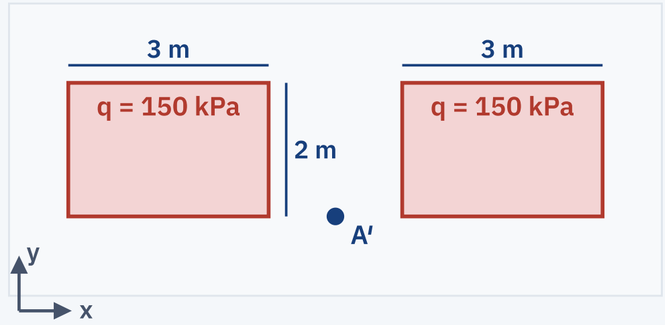
EXAMPLE

Given

Rectangle 1	3×2 m, $q = 150$ kN/m², center $x = 0$
Rectangle 2	3×2 m, $q = 150$ kN/m², center $x = 5$ m ($y = 0$)
Point	**A′:** $x = 2.5$ m, $y = -1$ m; $z = 4$ m (also 1; 2; 6; 8; 12 m)

REQUIRED

For each rectangle and the total $\Delta\sigma_z$; the depth profile.



Solution (1/2)

SOLUTION

1 Rectangle 2: corners marked relative to A′ (5 m between centers ≥ $(L_1 + L_2)/2$ = 3 m: no overlap)

Corner (x; y)	a	b	m	n	sign
(6.5; 1.0)	4.0	2.0	1.00	0.50	+
(6.5; -1.0)	4.0	0.0	1.00	0.00	0 (b = 0)
(3.5; 1.0)	1.0	2.0	0.25	0.50	–
(3.5; -1.0)	1.0	0.0	0.25	0.00	0 (b = 0)

2 Superposition

$$\Delta\sigma_{z(2)} = 150\,[I_3(1;\,0.5) - I_3(0.25;\,0.5)] = 10.896\ \text{kPa}$$

$$\Delta\sigma_z = \Delta\sigma_{z(1)} + \Delta\sigma_{z(2)} = 10.896 + 10.896 = \mathbf{21.793}\ \text{kPa}$$

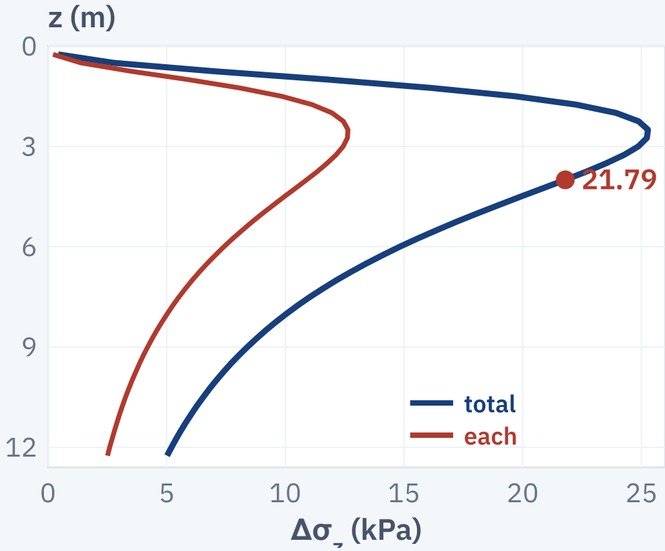
A′ is exactly at the midpoint of the two rectangles (x = 2.5 m; 1 m from each): rectangle 2 is the **mirror image** of rectangle 1 and gives the same stress. Total **$\Delta\sigma_z$ = 21.79 kPa**.

Solution (2/2)

SOLUTION

z (m)	Rectangle 1	Rectangle 2	Total
1	5.877	5.877	**11.754**
2	11.965	11.965	**23.930**
4	10.896	10.896	**21.793**
6	7.413	7.413	**14.826**
8	5.028	5.028	**10.056**
12	2.603	2.603	**5.205**

Total stress at z = 4 m is **21.79 kPa**; since the two rectangles are symmetric, the total at any depth = 2 × a single rectangle. The largest total is 25.28 kPa at z ≈ 2.50 m.



Check the solution in GeoTerva ↗

Summary

Key points

Corner: $\Delta\sigma_z = q\, I_3(m, n)$, $m = B/z$, $n = L/z$; for a large area $I_3 \to 1/4$. **Any point:** signed four-corner rectangle, $\Delta\sigma_z = q\, \Sigma \pm I_3$

Example (A′; z = 4 m): 4 m × 2 m − 1 m × 2 m: $I_3(0.5; 1) = 0.12018$; $I_3(0.25; 0.5) = 0.04753$; **$\Delta\sigma_z$ = 150 (0.07264) = 10.90 kPa**

Center: $\Delta\sigma_z = q\, I_4$; square, z = b: $I_4 = 4\, I_3(1, 1) = 0.701$; 3 × 2 m rectangle, z = 4 m: **22.98 kPa**

Location (z = 4 m): center 22.98, mid-sides 20.26 and 18.08, corner 16.06, A′ 10.90, far point 3.91 kPa; the stress is not zero outside either

Shape: at z = 4b, a square gives 0.108 q, a 10:1 rectangle 0.303 q, a strip 0.306 q. **Two rectangles:** the stresses are summed (10.90 + 10.90 = 21.79 kPa)

References

Das, B.M. (2018). Principles of Geotechnical Engineering, 9th ed. Cengage Learning.

Boussinesq, J. (1883). Application des Potentiels à l'Étude de l'Équilibre et du Mouvement des Solides Élastiques. Gauthier-Villars, Paris.

NOTE

All numerical examples have been generated with the module's own calculation engine; the module yields the same values. The module opens with a 3 m × 2 m rectangle (q = 150 kN/m²) and point A′ (x = 2.5 m; y = −1 m; z = 4 m).

GeoTerva

Online geotechnical analysis

From theory to practice

Enter your rectangles (q, L, B, centre), point A and the depth in GeoTerva; the module provides the four corner terms, the $\Delta\sigma_z$ value, the isostress contours, the depth profile, the step-by-step solution and the report together.

Open the module → **Theory page →**

Open on your phone
The example modules need no registration.

If you would like to use them in your lectures: info@geoterva.com