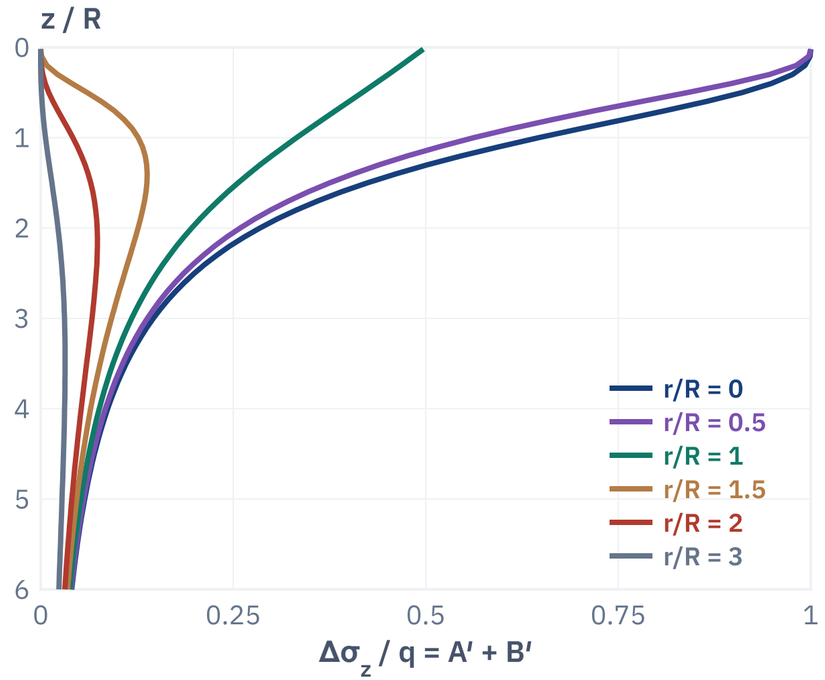
Geotechnical lecture presentation · Stresses in the soil mass

Stress under a circular load

Lecture: vertical stress under a circular load; closed-form expression on the axis, A′ + B′ off the axis; worked examples solved by hand.

On the axis | A′ + B′ | Polar integral | Outside the load | Multiple circles



Introduction

What will we learn in this lecture?

The vertical stress increase beneath a **circular area** with a uniformly distributed load (tank, silo, circular foundation): $\Delta\sigma_z$

On the axis of the circle closed-form expression: $\Delta\sigma_z = q\{1 - 1/[(R/z)^2 + 1]^{3/2}\}$

Eccentric, beneath the edge and beside it: $\Delta\sigma_z = q\,(A' + B')$; $A' + B'$ depends only on z/R and r/R

The evaluation of the Boussinesq integral **in polar coordinates at point A**: the ρ integral in closed form, the φ integral numerically

That outside the load the stress **first increases and then decreases**; that the stresses of several circles are **summed**

LECTURE FLOW

Each topic is covered in **three steps**:

1 · **Theory**: formula and rationale

2 · **Problem statement**: given and required

3 · **Solution**: step by step, with numbers

Example circle: R = 3 m, q = 100 kN/m²; point A is at a distance r = 4.5 m from the axis of the circle and at a depth z = 3 m (outside the loaded area).

Uniformly loaded circular area

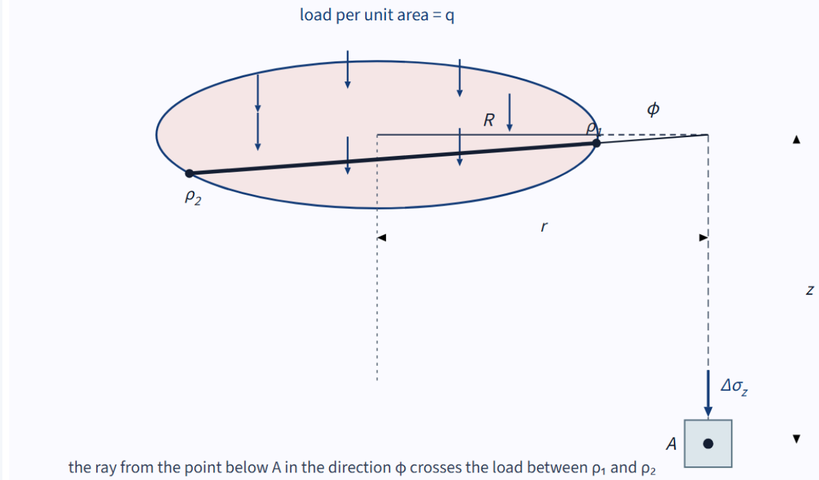
THEORY

Flexible circular area of radius **R** carrying a uniform pressure **q** (tank, silo, circular foundation)

The soil is a **homogeneous, isotropic and linearly elastic** semi-infinite medium; $\Delta\sigma_z$ is only the increase produced by the load

The problem is **axially symmetric**: the stress depends only on the horizontal distance from the center **r** and the depth **z**, not on the direction

Under the centre (r = 0) there is a **closed-form expression**; outside the axis, under and beside the edge, Ahlvin and Ulery (1962) evaluated the integral and tabulated it as **A′ and B′**



Beneath the center of the circle: closed-form solution

THEORY

$$\Delta\sigma_z = q\left\{1 - \frac{1}{\left[(R/z)^2 + 1\right]^{3/2}}\right\}$$

At the surface ($z \to 0$) $\Delta\sigma_z = q$; it **decreases monotonically** with depth and tends to zero at great depth

The stress depends only on the **R/z** ratio: for the same z/R a large and a small circle give the same $\Delta\sigma_z/q$ value

INFLUENCE DEPTH

$\Delta\sigma_z/q$	z/R
0.50	1.30
0.20	2.50
0.10	3.71
0.05	5.36

At a depth of 3.7R, the stress drops to only one tenth of its surface value.

$\Delta\sigma_z = q\,(A' + B')$

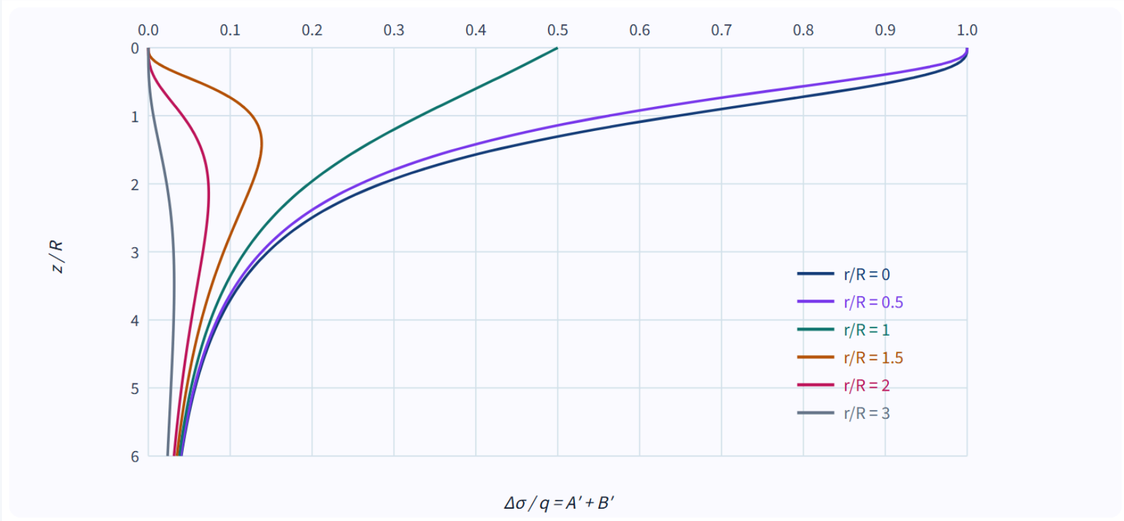
THEORY

$$\Delta\sigma_z = q\,(A' + B')$$

A′ and B′ are two parts that depend on the ratios **z/R** and **r/R**; they arise from the decomposition of the integral, and for $\Delta\sigma_z$ only their **sum** is needed

On the axis (r = 0) it reduces to a total closed-form expression; **under the edge** (r/R = 1) it starts from 0.5 at the surface

Beside the load the value starts from zero, first increases, then decreases



Polar integral about the point

THEORY

No interpolation is made between tables: the Boussinesq solution is integrated directly over the circle in the **ρ, φ** polar coordinates centered on the point below point A

In each φ direction, the ray intersects the loaded circle between **ρ_1** and **ρ_2** (ρ_1 = 0 if A is under the load); the load element is q ρ dρ dφ

$$d\sigma_z = \frac{3q}{2\pi} \frac{z^3 \rho}{(\rho^2 + z^2)^{5/2}} d\rho \, d\varphi$$

$$\int_{\rho_1}^{\rho_2} \frac{3z^3 \rho \, d\rho}{(\rho^2 + z^2)^{5/2}} = \frac{z^3}{(\rho_1^2 + z^2)^{3/2}} - \frac{z^3}{(\rho_2^2 + z^2)^{3/2}}$$

A′ + B′ and numerical sum

THEORY

$$A' + B' = \frac{1}{2\pi} \int_0^{2\pi} \left[\frac{z^3}{(\rho_1^2 + z^2)^{3/2}} - \frac{z^3}{(\rho_2^2 + z^2)^{3/2}} \right] d\varphi$$

$$\rho_{1,2} = -r \cos\varphi \mp \sqrt{R^2 - r^2 \sin^2\varphi}$$

The integrand of the remaining φ integral is continuous and is summed numerically over **2880 directions**; at r = 0, ρ_1 = 0 and ρ_2 = R, and the expression returns to the **closed-form solution**

A′ + B′ VALUES

z/R	r/R	A′ + B′
1.0	1.0	0.33224
1.0	1.5	0.12664
2.0	0.4	0.26781
2.0	2.0	0.07334
0.5	0.8	0.64625
4.0	1.5	0.06487

Superposition

THEORY

Each circle is defined by its own **q, R** and its centre on the ground surface (x_L, y_L) (at most eight circles in the module)

The two circles **may touch but cannot overlap**: the distance between centers is at least $R_1 + R_2$

Since the soil is elastic, the stress of each circle is found with **its own radial distance r from point A to that circle's center**, and the stresses are summed

$$\Delta\sigma_z = \sum_k q_k \, (A' + B')_k \qquad r_k = \sqrt{(x - x_{L,k})^2 + (y - y_{L,k})^2}$$

Problem statement

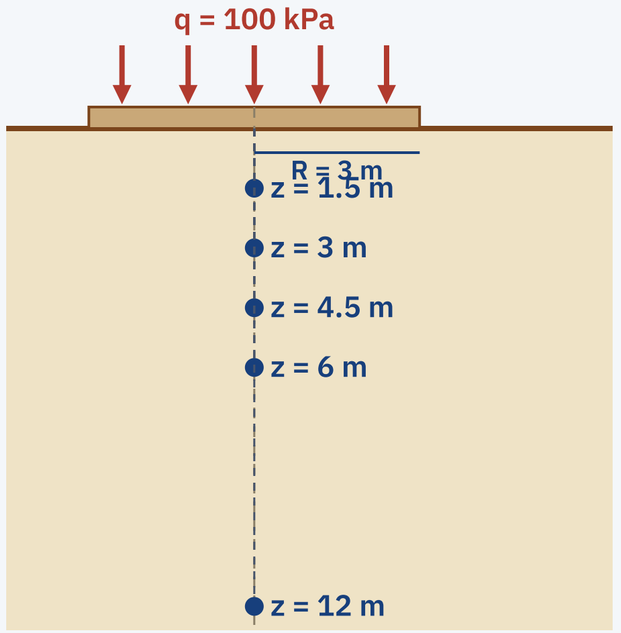
EXAMPLE

REQUIRED

Closed-form **$\Delta\sigma_z$** (first detailed at z = 1.5 m), **A′ + B′**; the depth at which $\Delta\sigma_z = 0.10$ q.

Given

Circle	**flexible, $R = 3$ m, $q = 100$ kN/m²**
Points	**on the axis of the circle ($r = 0$): $z = 1.5; 3; 4.5; 6; 12$ m**



Solution (1/3)

SOLUTION

1 Ratios

$$R/z = 3/1.5 = 2 \qquad (R/z)^2 + 1 = 4 + 1 = \mathbf{5}$$

2 Exponent and inverse value

$$5^{3/2} = 11.1803 \qquad 1/11.1803 = 0.08944$$

3 Vertical stress

$$\Delta\sigma_z = q\,(1 - 0.08944) = 100\,(0.91056) = \mathbf{91.06}\ \mathrm{kPa}$$

At z = 1.5 m depth, **A′ + B′ = 0.91056** and **$\Delta\sigma_z$ = 91.06 kPa**: 91.1% of the surface load is transmitted.

Solution (2/3)

SOLUTION

z (m)	z/R	$(R/z)^2 + 1$	$[\ldots]^{3/2}$	$1 - 1/[\ldots]^{3/2}$	$\Delta\sigma_z$ (kPa)
1.5	0.50	5.000	11.1803	0.91056	**91.06**
3.0	1.00	2.000	2.8284	0.64645	**64.64**
4.5	1.50	1.444	1.7360	0.42397	**42.40**
6.0	2.00	1.250	1.3975	0.28446	**28.45**
12.0	4.00	1.062	1.0952	0.08692	**8.69**

On the axis the stress decreases in one direction: **91.1 → 8.7 kPa** (z = 1.5 m → 12 m). The z = 1.5 m row is the hand-solved calculation; each row is found with the same expression. At z = 3 m the stress is about 65% of the surface load, and at z = 12 m it is 9%.

Solution (3/3)

SOLUTION

4 Depth at which $\Delta\sigma_z = 0.10\ q$

$$\left[(R/z)^2 + 1\right]^{3/2} = 1/0.90 \;\Rightarrow\; (R/z)^2 = 1.0728 - 1 = 0.0728$$

$$z/R = 1/\sqrt{0.0728} = \mathbf{3.707} \;\Rightarrow\; z = 3\,(3.707) = \mathbf{11.12}\ \mathrm{m}$$

The depth at which $\Delta\sigma_z = 0.10\ q$ is **11.12 m** (z/R = 3.707): at this depth the stress on the axis has fallen to one tenth of the surface load. The half value (0.50 q) is at z/R = 1.305, i.e. 3.91 m.

Check the solution in GeoTerva ↗

Problem statement

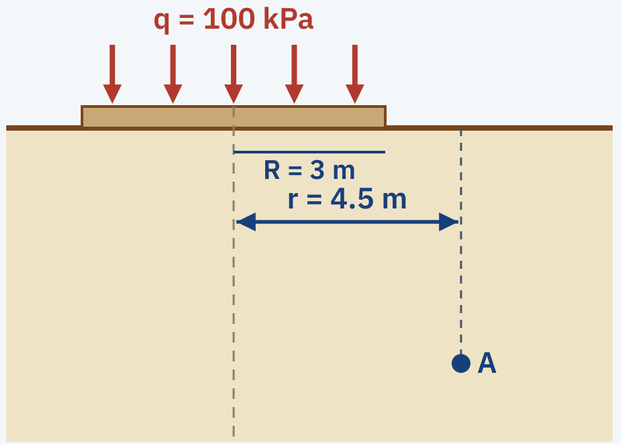
EXAMPLE

Given

Circle	**flexible, $R = 3$ m, $q = 100$ kN/m²**
Point A	**$r = 4.5$ m from the center ($r/R = 1.5$; outside the load), $z = 3$ m ($z/R = 1$)**
Method	**polar integral: first by hand, the $N = 36$ direction; then the module's 2880 directions**

REQUIRED

A′ + B′ and **$\Delta\sigma_z$**; the effect of the number of directions on the result.



Solution (1/3)

SOLUTION

1 Distances at which the ray intersects the circle

$$\rho_{1,2} = -r\cos\varphi \mp \sqrt{R^2 - r^2\sin^2\varphi} = -4.5\cos\varphi \mp \sqrt{9 - 20.25\sin^2\varphi}$$

2 In which directions do the rays hit the circle?

$$R^2 - r^2\sin^2\varphi \geq 0 \;\Rightarrow\; |\sin\varphi| \leq R/r = 0.6667 \;\text{ and }\; \rho > 0 \;\Rightarrow\; \cos\varphi < 0$$

Since A lies outside the circle, only directions facing the circle contribute: **138.19° < φ < 221.81°**.

The ray strikes the circle only in the range **138.19° < φ < 221.81°**. If **N = 36** directions are taken, φ_i = 5°, 15°, 25°, ...; of these only **8** lie in this range: 145°, 155°, 165°, 175° and their symmetric counterparts 185°, 195°, 205°, 215°.

Solution (2/3)

SOLUTION

3 Contribution of one ray and average

$$F(\rho) = \frac{z^3}{(\rho^2 + z^2)^{3/2}} = \frac{27}{(\rho^2 + 9)^{3/2}} \qquad g(\varphi) = F(\rho_1) - F(\rho_2)$$

$$A' + B' = \frac{1}{N} \sum_{i=1}^{N} g(\varphi_i) \qquad \varphi_i = \left(i - \tfrac{1}{2}\right) \frac{360^\circ}{N}$$

φ (°)	cos φ	$R^2 - r^2 \sin^2\varphi$	ρ_1	ρ_2	$F(\rho_1)$	$F(\rho_2)$	$g = F(\rho_1) - F(\rho_2)$
145	−0.8192	2.3380	2.1571	5.2152	0.5352	0.1240	**0.4112**
155	−0.9063	5.3832	1.7582	6.3986	0.6422	0.0765	**0.5657**
165	−0.9659	7.6435	1.5820	7.1114	0.6921	0.0587	**0.6334**
175	−0.9962	8.8462	1.5086	7.4571	0.7131	0.0520	**0.6611**

The 185°, 195°, 205°, 215° rays are the symmetric counterparts of 175°, 165°, 155°, 145°: they give the same g values. g(145°) = 0.4112 and g(175°) = 0.6611.

Solution (3/3)

SOLUTION

4 Total (N = 36)

$$\sum g = 2\,(0.4112 + 0.5657 + 0.6334 + 0.6611) = 4.5427$$

$$A' + B' \approx \frac{4.5427}{36} = 0.12619 \quad \Rightarrow \quad \Delta\sigma_z \approx 100\,(0.12619) = \mathbf{12.62}\ \text{kPa}$$

5 The result converges as the number of directions increases

N	6	12	24	36	72	180	720	2880
$\Delta\sigma_z$ (kPa)	16.856	10.556	13.200	12.619	12.541	12.688	12.667	12.664

The module uses **2880 directions**: **A′ + B′ = 0.12664**, **$\Delta\sigma_z$ = 12.664 kPa**. The tabulated value for the same point is 0.12663; the manual result with 36 directions (12.62) differs from this by only 0.4%.

Check the solution in GeoTerva ↗

Problem statement

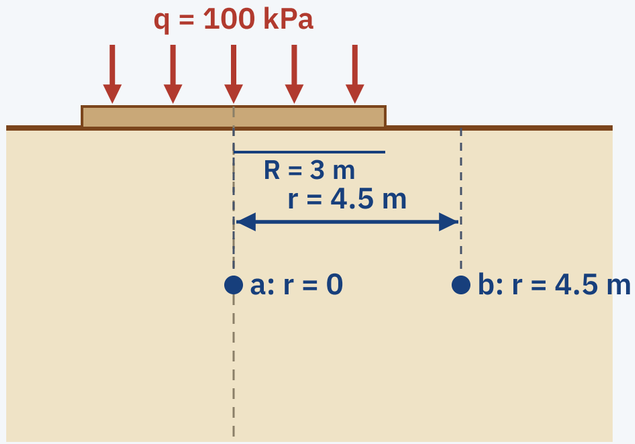
EXAMPLE

REQUIRED

$\Delta\sigma_z$ on both lines; the **maximum stress** and its depth on line (b).

Given

Circle	**flexible,** $R = 3$ **m,** $q = 100$ **kN/m²**
Points	**(a) axis,** $r = 0$**; (b)** $r = 4.5$ **m** $(r/R = 1.5)$
Depths	$z = 1.5; 3; 4.5; 6; 12$ **m**



Solution (1/3)

SOLUTION

z (m)	z/R	(a) A′ + B′	(a) $\Delta\sigma_z$ (kPa)	(b) A′ + B′	(b) $\Delta\sigma_z$ (kPa)
1.5	0.50	0.91056	91.06	0.06044	**6.04**
3.0	1.00	0.64645	64.64	0.12664	**12.66**
4.5	1.50	0.42397	42.40	0.13793	**13.79**
6.0	2.00	0.28446	28.45	0.12646	**12.65**
12.0	4.00	0.08692	8.69	0.06487	**6.49**

Axis: 91.1 → 8.7 kPa, decreasing in one direction. **Outside the load (r = 4.5 m):** 6.04 → 13.79 → 6.49 kPa; first it **increases**, then decreases: the load spreads laterally with depth.

Each row is a 2880-direction A′ + B′ value; the row z = 3 m in column (b) is the result of Example 2.

Solution (2/3)

SOLUTION

1 A′ + B′ along the r/R = 1.5 line, versus depth

z/R	z (m)	A′ + B′	$\Delta\sigma_z$ (kPa)
1.00	3.00	0.12664	**12.66**
1.20	3.60	0.13569	**13.57**
1.41	4.23	0.13833	**13.83**
1.60	4.80	0.13670	**13.67**
1.80	5.40	0.13238	**13.24**
2.00	6.00	0.12646	**12.65**

Along the line r/R = 1.5, the **maximum stress is at a depth of z ≈ 4.23 m** (z/R = 1.41) and equals **13.83 kPa**: outside the loaded area the stress reaches its maximum value at depth and then decreases.

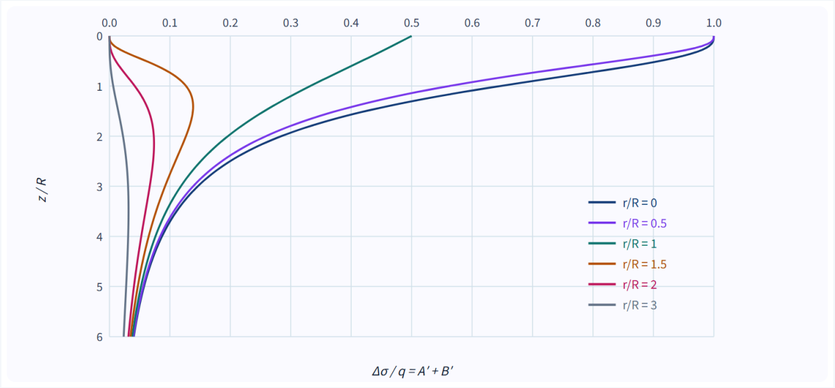
Solution (3/3)

SOLUTION

2 Maximum outward value

r/R	largest A′ + B′	depth z/R
1.5	0.1383	1.41
2.0	0.0737	2.15
3.0	0.0317	3.48

As the point moves away from the circle, the maximum stress is **deeper and smaller**: for r/R = 1.5, z/R = 1.41; for r/R = 3, z/R = 3.48.



Check the solution in GeoTerva ↗

Problem statement

EXAMPLE

Given

Table	$A' + B'$ **values:** $z/R = 0.25$ **…** 6**;** $r/R = 0$ **…** 3
Another circle	$R = 5$ **m,** $q = 120$ **kN/m²; point** $r = 7.5$ **m,** $z = 5$ **m**

REQUIRED

A′ + B′ table; in the second circle **Δσ$_z$** (from the table) and by direct calculation.

Solution (1/2)

SOLUTION

z/R \ r/R	0.0	0.5	1.0	1.5	2.0	3.0
0.25	0.9857	0.9658	0.4596	0.0142	0.0016	0.0001
0.50	0.9106	0.8396	0.4175	0.0604	0.0105	0.0010
1.00	0.6464	0.5622	0.3322	0.1266	0.0418	0.0062
1.50	0.4240	0.3750	0.2562	0.1379	0.0650	0.0146
2.00	0.2845	0.2589	0.1960	0.1265	0.0733	0.0225
3.00	0.1462	0.1384	0.1181	0.0919	0.0666	0.0309
4.00	0.0869	0.0840	0.0761	0.0649	0.0526	0.0311
6.00	0.0403	0.0396	0.0377	0.0349	0.0314	0.0236

The **r/R = 0** column is the closed-form solution; at **r/R = 1** the value approaches 0.5 near the surface (0.4596 at z/R = 0.25); in the **r/R ≥ 1.5** columns the value first increases and then decreases. Every number in the table is a 2880-direction integral.

Solution (2/2)

SOLUTION

1 Ratios

$$\frac{z}{R} = \frac{5}{5} = 1.0 \qquad \frac{r}{R} = \frac{7.5}{5} = 1.5$$

2 Coefficient and stress from the table

$$A' + B' \,(z/R = 1;\; r/R = 1.5) = 0.12664$$

$$\Delta\sigma_z = q\,(A' + B') = 120\,(0.12664) = \mathbf{15.20}\ \mathrm{kPa}$$

$\Delta\sigma_z/q$ depends only on **z/R** and **r/R**: for R = 5 m in the circle, the same ratios give the coefficient from the R = 3 m circle in Example 2 (**0.12664**); the stress scales only with q (15.20 kPa).

Check the solution in GeoTerva ↗

Problem statement

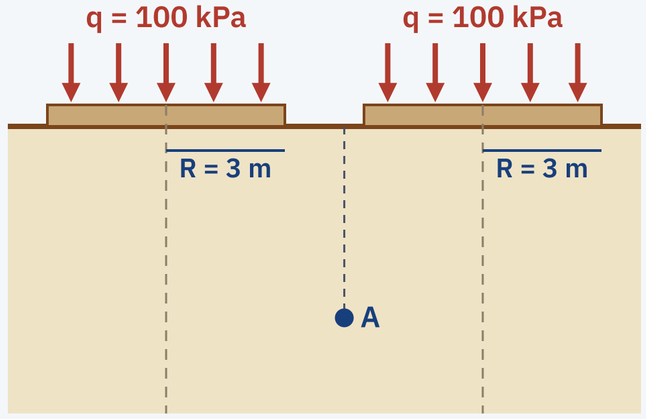
EXAMPLE

Given

Circle 1	$R = 3$ m, $q = 100$ kN/m², center $x = 0$
Circle 2	$R = 3$ m, $q = 100$ kN/m², centre $x = 8$ m
Point A	$x = 4.5$ m, $y = 0$; $z = 3$ m (also 1.5; 4.5; 6; 12 m)

REQUIRED

Separate and combined $\Delta\sigma_z$ for the two circles; depth profile.



Solution (1/2)

SOLUTION

1 Overlap check (circles do not overlap; 2 m between them) and spacings

$$d = 8\ \text{m} \ \geq R_1 + R_2 = 6\ \text{m}$$

$$r_1 = 4.5\ \text{m}\ (r_1/R = 1.5) \qquad r_2 = 8 - 4.5 = 3.5\ \text{m}\ (r_2/R = 1.1667)$$

2 Influence factor and stress for each circle (z/R = 1)

$$A' + B'_{(1)} = 0.12664 \ \Rightarrow\ \Delta\sigma_{z(1)} = 100\,(0.12664) = 12.664\ \text{kPa}$$

$$A' + B'_{(2)} = 0.25049 \ \Rightarrow\ \Delta\sigma_{z(2)} = 100\,(0.25049) = 25.049\ \text{kPa}$$

3 Superposition

$$\Delta\sigma_z = \Delta\sigma_{z(1)} + \Delta\sigma_{z(2)} = 12.664 + 25.049 = \mathbf{37.714}\ \text{kPa}$$

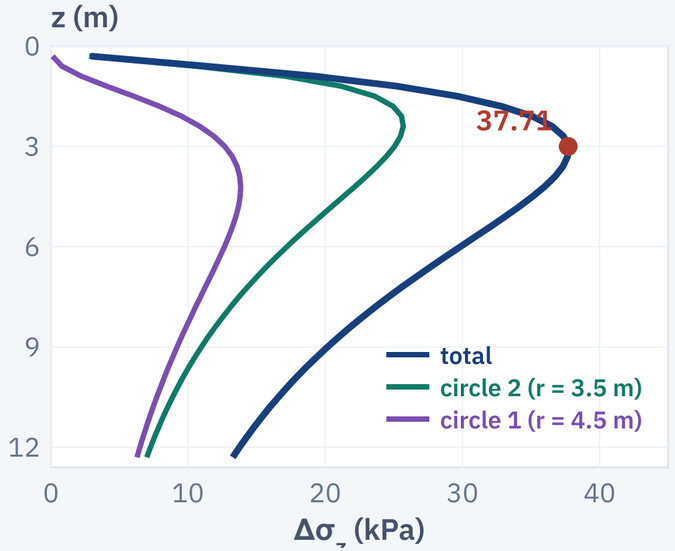
A is closer to the second circle (3.5 m and 4.5 m): **circle 2 gives 66% of the total**. Total $\Delta\sigma_z$ = **37.71 kPa**; the value for a single circle was 12.66 kPa.

Solution (2/2)

SOLUTION

z (m)	z/R	$r_1/R = 1.5$	$r_2/R = 1{,}167$	$\Delta\sigma_{z(1)}$	$\Delta\sigma_{z(2)}$	Total
1.5	0.50	0.06044	0.23608	6.04	23.61	**29.65**
3.0	1.00	0.12664	0.25049	12.66	25.05	**37.71**
4.5	1.50	0.13793	0.21324	13.79	21.32	**35.12**
6.0	2.00	0.12646	0.17204	12.65	17.20	**29.85**
12.0	4.00	0.06487	0.07258	6.49	7.26	**13.75**

The total stress at z = 3 m is **37.71 kPa**; the contribution of circle 2 decreases with depth while that of circle 1 increases up to 4.5 m. The result is the sum of the values found with each circle's own distance r.



Check the solution in GeoTerva ↗

Problem statement

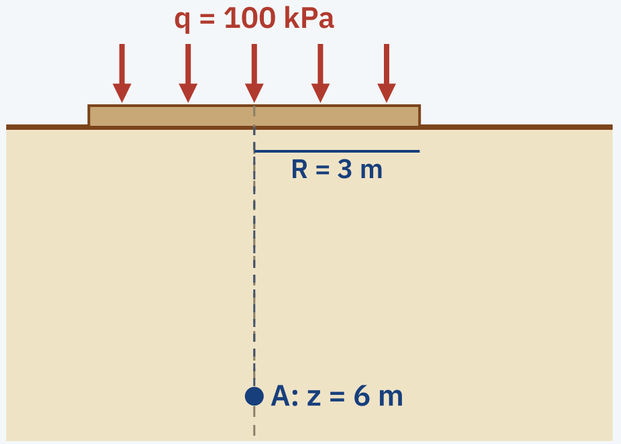
EXAMPLE

Given

Circle	$R = 3$ m, $q = 100$ kN/m²; total load $Q = q\pi R^2 = 2827.4$ kN
Point load	at the center of the same Q circle: $\Delta\sigma_z = Q\, I_1/z^2$, $I_1 = \frac{3}{2\pi}[(r/z)^2 + 1]^{-5/2}$
Points	on the axis at ($r = 0$) and at $r = 2R$; $z/R = 1; 2; 3; 5; 10$

REQUIRED

Ratio of the two solutions; beyond what depth is the point-load approximation acceptable?



Solution (1/2)

SOLUTION

1 Total load

$$Q = q\pi R^2 = 100\,\pi\,(3)^2 = \mathbf{2{,}827.4}\ \mathrm{kN}$$

2 Point load (Boussinesq), on the axis r = 0

$$\Delta\sigma_z = \frac{Q}{z^2}\,\frac{3}{2\pi} = \frac{2{,}827.4}{36}\,(0.4775) = \mathbf{37.50}\ \mathrm{kPa}$$

3 Circle (closed-form expression, z = 6 m)

$$\Delta\sigma_z = 100\left(1 - \frac{1}{(0.25+1)^{3/2}}\right) = \mathbf{28.45}\ \mathrm{kPa}$$

At depth z = 2R the point-load approximation gives the stress **1.318 times** larger (31.8%): 28.4 kPa instead of 37.5 kPa.

Solution (2/2)

SOLUTION

r/R	z/R	Circle $\Delta\sigma_z$ (kPa)	Point load $\Delta\sigma_z$ (kPa)	Point load / circular
0	1	64.645	150.000	**2.320**
0	2	28.446	37.500	**1.318**
0	3	14.619	16.667	**1.140**
0	5	5.713	6.000	**1.050**
0	10	1.481	1.500	**1.013**
2	1	4.181	2.683	**0.642**
2	2	7.334	6.629	**0.904**
2	3	6.663	6.647	**0.997**
2	5	4.048	4.140	**1.023**
2	10	1.346	1.360	**1.010**

On the axis at z/R = 1, the point load overestimates the stress by a factor of **2.32**; at z = 5R the difference is 5.0%, and at z = 10R it drops to 1.3%. At r = 2R the approximation is **low** near the surface (0.64), while at z ≥ 3R the difference does not exceed 3%.

Check the solution in GeoTerva ↗

Summary

Key points

On the axis: $\Delta\sigma_z = q\{1 - 1/[(R/z)^2 + 1]^{3/2}\}$; for z = 1.5 m, 91.06 kPa (A′ + B′ = 0.91056); the depth at which $\Delta\sigma_z$ = 0.10 q is **3.71R** (11.12 m)

General point: $\Delta\sigma_z = q\,(A' + B')$; A′ + B′ depends only on z/R and r/R. **Example (r = 4.5 m; z = 3 m):** A′ + B′ = 0.12664, **$\Delta\sigma_z$ = 12.664 kPa**

Calculation: the Boussinesq integral at point A in polar coordinates; ρ closed-form, φ numerical (2880 directions). By hand with 36 directions: 12.62 kPa

Outside the load (r/R = 1.5) the stress first increases: maximum value 13.83 kPa at z ≈ 4.23 m; below the edge, 0.5 q at the surface

Two circles: the stresses are found using each circle's own distance r and summed: 12.66 + 25.05 = 37.71 kPa. The **point load** approximation is in error near the load (×1.32 at z = 2R), and is acceptable at z ≥ 5R

References

Das, B.M. (2018). Principles of Geotechnical Engineering, 9th ed. Cengage Learning.

Ahlvin, R.G. and Ulery, H.H. (1962). Tabulated values for determining the complete pattern of stresses, strains, and deflections beneath a uniform circular load on a homogeneous half space. Highway Research Bulletin 342, 1–13.

NOTE

All numerical examples were produced with the module's own calculation engine; the module gives the same values. The module opens with R = 3 m, q = 100 kN/m² circular load and point A (r = 4.5 m; z = 3 m).

GeoTerva

Online geotechnical analysis

From theory to practice

Enter your circles (q, R, center), point A and the depth in GeoTerva; the module provides A′ + B′, the $\Delta\sigma_z$ value, isostress contours, the depth profile, the step-by-step solution and the report together.

Open the module →

Theory page →

Open on your phone
The example modules need no registration.

If you would like to use them in your lectures: info@geoterva.com