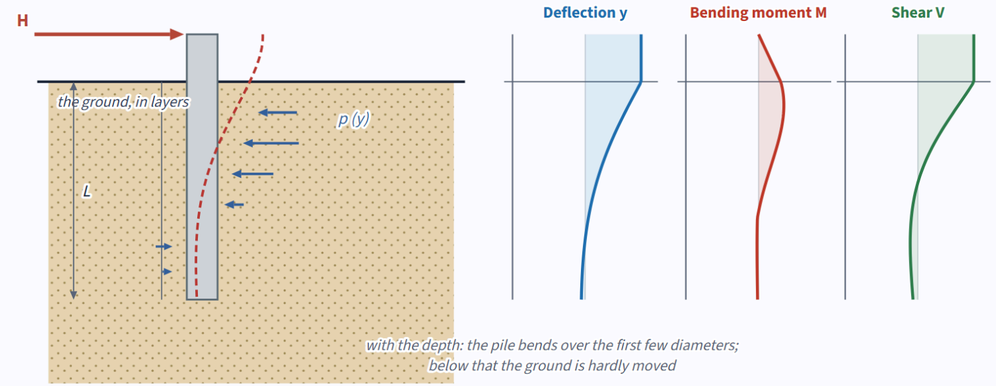
Geotechnical lecture presentation · Piles

Laterally loaded pile: p–y method

Lecture: analysis of a laterally loaded pile as a beam on springs, sand and clay p–y curves, cyclic loading, axial load and head conditions; a worked example for each topic.

- Beam on springs
- Sand and clay p–y curves
- Cyclic loading
- Initial conditions
- Worked examples



Introduction

What will we learn in this lecture?

- how to set up a laterally loaded pile as a **beam on springs**: equation, sign conventions, end conditions
- **Elastic solutions** (Matlock–Reese: sand, n_h; Hetényi: clay, k) computed by hand
- to construct the API (Reese–Cox–Koop) p–y curves for **sand** and the Matlock p–y curves for **clay**, static and cyclic, point by point
- Knowing **what the numerical solution (finite elements, Newton) does** and how to read the results
- To interpret the effects of the **axial load** (P–δ), the **head condition** and the **free length**
- To find the **load** corresponding to an allowable displacement or to the moment capacity

LECTURE FLOW

Each topic is covered in **three steps**:

1 · **Theory**: what is the rule, why is it so?

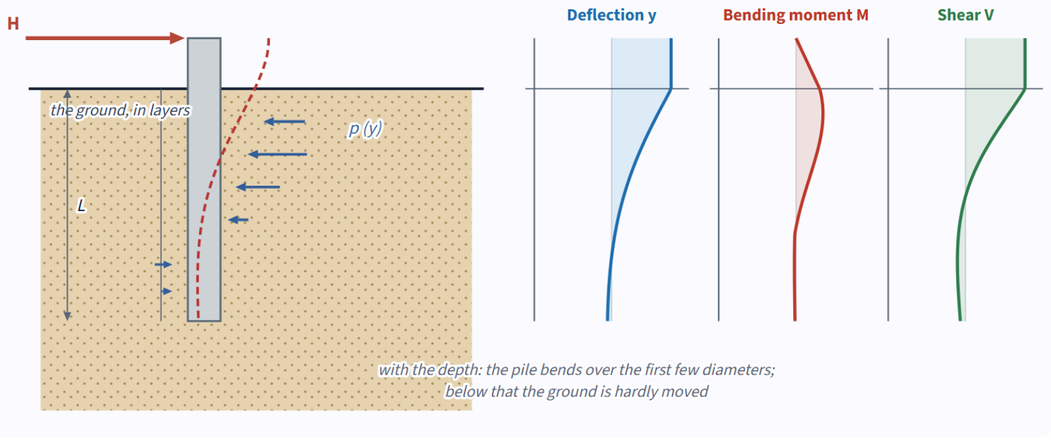
2 · **Problem statement**: given and required

3 · **Solution**: step by step, with numbers

Example pile: a concrete pile 0.60 m in diameter and 14 m long; in sand–clay–sand soil, a lateral load of 120 kN at ground level. The module also opens with this model.

The load reaches the soil only through lateral movement

THEORY



A laterally loaded pile does not fail like a standing foundation: the load is transmitted to the soil **only through the lateral movement of the pile**; the soil reaction grows with movement and stops growing when the soil yields. The pile bends like a beam over the first **few diameters** of soil; deeper down the soil hardly moves.

Ultimate load (Broms): the load at which the soil fails for a short pile and the section fails for a long pile. It gives capacity but not the displacement or moment at the working load.

Elastic subgrade modulus (Matlock–Reese, Davisson–Gill, Heténi): soil as elastic springs; closed-form solution.

p–y method (Matlock, Reese et al.): the springs are **nonlinear**; p starts rigid with respect to y, softens and ends at the ultimate resistance of the soil. It is solved numerically; layered soil, from working load to near-failure load.

Equation, signs and boundary conditions

THEORY

A pile is a beam, the soil along it a row of springs; each spring acts at its own depth and independently of the others (Winkler). z is downward from the surface, y is the displacement in the direction of the load:

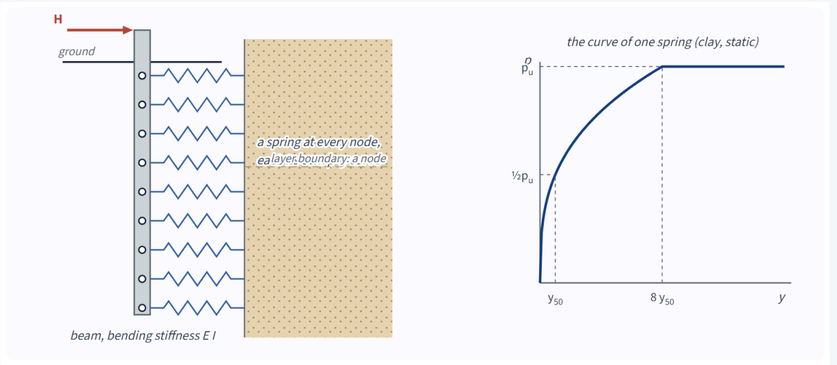
$$EI\,\frac{d^4y}{dz^4} + P\,\frac{d^2y}{dz^2} + p(y,z) = 0$$

$$\theta = \frac{dy}{dz} \qquad M = EI\frac{d^2y}{dz^2} \qquad V = EI\frac{d^3y}{dz^3} + P\theta \qquad \frac{dV}{dz} = -p$$

p is positive when it resists displacement. For spring stiffness k, p = k y; **k = k_h D** (kN/m², force per unit length and unit displacement).

Free tip: M = 0, V = 0. **Head**: if free, H and M_g are given; if fixed against rotation, the slope is zero and the restraining moment is part of the solution.

All the equation needs is the soil law p(y, z): constant k·y in the elastic case, p–y curves in the nonlinear case.



Sand: $k = n_h z$

THEORY

In sand the modulus of subgrade reaction is taken directly proportional to depth: **$k = n_h z$**. The pile has a single **characteristic length**; for load Q and moment M_g at the ground surface, the displacement and moment are:

$$T = \left(\frac{EI}{n_h} \right)^{1/5}$$

$$y = A_y \frac{QT^3}{EI} + B_y \frac{M_g T^2}{EI} \qquad M = A_m QT + B_m M_g$$

The A and B coefficients depend on Z = z/T (at the side). The pile is considered **long**: L/T is large, the tip does not move.

The module reproduces these coefficients with its own calculation engine: A_y = 2.4292.

COEFFICIENTS (LONG PILE)

Z	A_y	A_θ	A_m	B_y	B_θ	B_m
0	2,429	−1,619	0.000	1,619	−1,747	1,000
0.5	1,640	−1,500	0.458	0.871	−1,250	0.974
1	0.961	−1,194	0.725	0.364	−0.789	0.851
1.4	0.543	−0.891	0.770	0.112	−0.480	0.686
2	0.142	−0.462	0.627	−0.070	−0.153	0.404
3	−0.075	−0.039	0.224	−0.089	0.057	0.059

A_m is largest at Z ≈ 1.4 (0.77).

Problem statement

EXAMPLE

Given

Pile	**HP 250 × 85, L = 25 m, E I = 25 461 kN·m²**
Soil	**sand, n_h = 12 000 kN/m³**
Load	**Q at the ground surface; M_g = 0, free head**
Coefficients	$A_y = 2.4292$ **(head),** $A_m = 0.772$ **(maximum)**

REQUIRED

Find the load Q and **the maximum moment** such that the head displacement is **8 mm**. Is the pile long?

Solution

SOLUTION

1 Characteristic length and pile length

$$T = \left(\frac{25\,461}{12\,000}\right)^{1/5} = \mathbf{1.1624}\ \text{m} \qquad \frac{L}{T} = \frac{25}{1.1624} = 21.5 \Rightarrow \text{long pile}$$

2 Load: $y = A_y\, Q\, T^3 / E\, I \Rightarrow Q = y\, E\, I / (A_y\, T^3)$

$$Q = \frac{0.008(25\,461)}{2.4292\,(1.1624)^3} = \mathbf{53.4}\ \text{kN}$$

3 Maximum moment: $M = A_m\, T\, Q$

$$M_{max} = 0.772\,(1.1624)(53.4) = \mathbf{47.9}\ \text{kN}\cdot\text{m}$$

For this load the module gives a head displacement of **8.00 mm** and a maximum moment of **47.9 kN·m**: the same as the hand calculation.

Check the solution in GeoTerva ↗

Clay: constant k (Hetényi)

THEORY

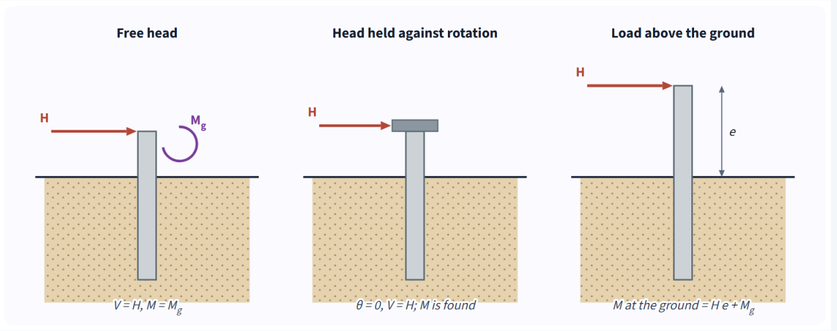
In clay the coefficient of subgrade reaction is taken **constant** with depth. k is the soil stiffness per unit length of pile; **β = (k / 4EI)$^{1/4}$**. For a long pile, Hetényi's closed-form solutions under a load H at the ground surface:

$$\text{free end:} \quad y_0 = \frac{2H\beta}{k} \qquad M_{max} = 0.3224\,\frac{H}{\beta} \quad \left(z = \tfrac{\pi}{4\beta}\right)$$

$$\text{restrained end:} \quad y_0 = \frac{H}{4EI\beta^3} \qquad M_{head} = -\frac{H}{2\beta}$$

The pile fixed against rotation at the head displaces **half as much** as the free-head pile, but the restraining moment at the head is **greater** than the maximum moment of the free-head pile.

The actual cap behavior lies between these two extremes; how close it is depends on the stiffness of the cap and the superstructure.



Problem statement

EXAMPLE

Given

Pile	**D = 0.60 m, E I = 190,852 kN·m² (example pile)**
Soil	**clay, k_h = 15,000 kN/m³ ⇒ k = k_h D = 9,000 kN/m²**
Load	**H = 120 kN, at the ground surface**
Pile	**long**

REQUIRED

Determine the pile-head displacement and moments for the **free-head** and **fixed-head** cases using Hetényi's closed-form solutions.

Solution (1/2)

SOLUTION

1 β

$$\beta = \left(\frac{k}{4EI}\right)^{1/4} = \left(\frac{9{,}000}{4(190{,}852)}\right)^{1/4} = \mathbf{0.3295}\ \mathrm{m}^{-1}$$

2 Free head

$$y_0 = \frac{2H\beta}{k} = \frac{2(120)(0.3295)}{9{,}000} = \mathbf{8.79}\ \mathrm{mm}$$

$$M_{max} = 0.3224\frac{H}{\beta} = 0.3224\frac{120}{0.3295} = \mathbf{117.4}\ \mathrm{kN{\cdot}m} \quad \left(z = \tfrac{\pi}{4\beta} = 2.38\ \mathrm{m}\right)$$

Solution (2/2)

SOLUTION

3 Fixed end

$$y_0 = \frac{H}{4EI\beta^3} = \frac{120}{4(190{,}852)(0.3295)^3} = \mathbf{4.39}\ \text{mm}$$

$$M_{head} = -\frac{H}{2\beta} = -\frac{120}{2(0.3295)} = \mathbf{-182.1}\ \text{kN·m}$$

It reduces the restrained-head displacement **by half** (8.79 → 4.39 mm), but the head moment (182.1 kN·m) is **1.55 times** the maximum moment of the free pile (117.4 kN·m). The module's elastic option gives the same numbers: free 8.79 mm and 117.4 kN·m, restrained 4.39 mm and 182.1 kN·m.

Check the solution in GeoTerva ↗

API RP 2A: Reese, Cox and Koop

THEORY

The soil near the surface is pushed in front of the pile as a **wedge**; deeper down, the soil **flows horizontally around** the pile. The ultimate resistance at depth z (σ′: the effective stress at that depth) is the smaller of the two:

$$p_{us} = (C_1 z + C_2 D)\,\sigma' \qquad p_{ud} = C_3 D\,\sigma'$$

$$p = A\,p_u \tanh\left(\frac{k\,z\,y}{A\,p_u}\right) \qquad A = \max\left(0.9;\ 3 - 0.8\tfrac{z}{D}\right)\ \text{(static)},\ A = 0.9\ \text{(cyclic)}$$

C_1, C_2, C_3 depend only on **φ′**; k, the initial modulus, is taken according to φ′ and water (at side).

At small y, the curve is the line **k z y** (the n_h = k case of the elastic solution); at large y, it ends at **A p_u**. A is 3.0 at the surface and 0.9 at a depth of 2.6D: static resistance is higher near the surface and absent under cyclic loading.

COEFFICIENTS VARYING WITH φ′

φ′	C_1	C_2	C_3	k (above water)	k (below water)
28°	1.60	2.41	22.5	5.4	5.4
30°	1.91	2.67	28.7	10.5	7.9
32°	2.28	2.95	36.8	20.7	11.9
34°	2.72	3.25	47.3	32.7	17.6
36°	3.24	3.59	61.2	46.3	24.8
38°	3.87	3.97	79.6	61.7	33.6
40°	4.62	4.38	104.1	78.9	44.0

k: MN/m³. For 32°, C_1 = 2.28; C_2 = 2.95; C_3 = 36.8.

Problem statement

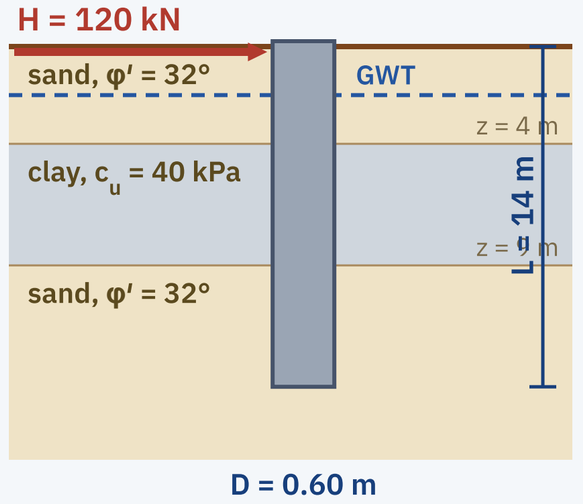
EXAMPLE

Given

Pile	**pile, circular section, D = 0.60 m, L = 14 m, concrete: E I = 190 852 kN·m²**
Soil	**sand 0–4 m (φ′ = 32°) · clay 4–9 m (c_u = 40 kPa) · sand 9–17 m (φ′ = 32°)**
Unit weight, water	**γ = 18, γ_{sat} = 19 kN/m³; groundwater at 2.0 m**
Depth	**z = 2.0 m (sand 1, GWT level); k = 20.7 MN/m³ (above water)**
Coefficients	$K_0 = 0.4,\ K_a = \tan^2(45^\circ - \varphi'/2)$

REQUIRED

With σ′, C_1, C_2, C_3, $\mathbf{p_u}$, A and y = 5 mm and 10 mm, find the **p** values.



Solution (1/2)

SOLUTION

1 Effective stress and angles (GWT = 2 m: moist from the surface up to z = 2)

$$\sigma' = 18(2) = \mathbf{36.0}\ \mathrm{kPa} \qquad \alpha = \frac{\varphi'}{2} = 16^\circ,\ \beta = 45^\circ + \frac{\varphi'}{2} = 61^\circ,\ K_a = \tan^2 29^\circ = 0.3073$$

2 Coefficients (φ′ = 32°)

$$C_1 = \frac{K_0 \tan\varphi' \sin\beta}{\tan(\beta-\varphi')\cos\alpha} + \frac{\tan^2\beta \tan\alpha}{\tan(\beta-\varphi')} + K_0 \tan\beta(\tan\varphi' \sin\beta - \tan\alpha) = \mathbf{2.281}$$

$$C_2 = \frac{\tan\beta}{\tan(\beta-\varphi')} - K_a = \mathbf{2.947}$$

$$C_3 = K_a(\tan^8\beta - 1) + K_0 \tan\varphi' \tan^4\beta = \mathbf{36.81}$$

Solution (2/2)

SOLUTION

3 Ultimate resistance: the lesser of the wedge and flow around the periphery

$$p_{us} = (2.281(2) + 2.947(0.6))(36.0) = 227.9 \text{ kN/m}$$

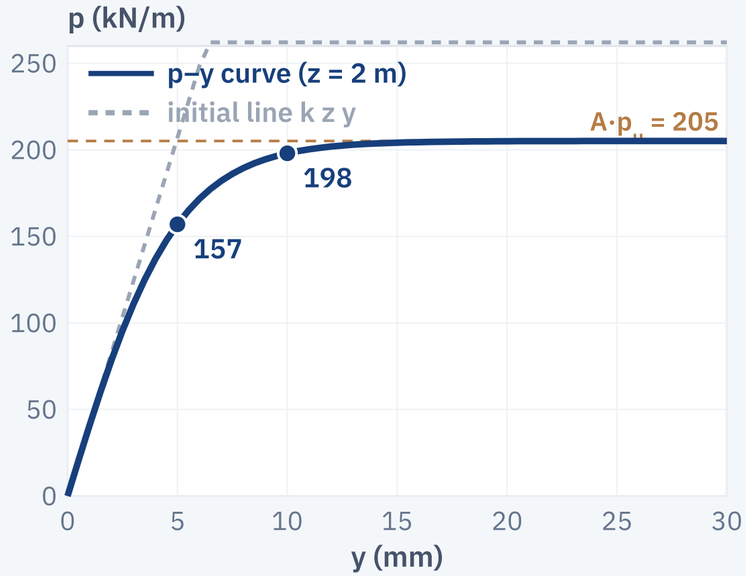
$$p_{ud} = 36.81(0.6)(36.0) = 795 \text{ kN/m} \Rightarrow p_u = \mathbf{227.9} \text{ kN/m}$$

4 A and the curve: A = max(0.9; 3 − 0.8 z/D) = max(0.9; 0.33) = 0.9; k z y / (A p_u)

$$y = 5 \text{ mm}: \; p = 0.9(227.9)\tanh\frac{20\,700(2)(0.005)}{0.9(227.9)}$$

$$= \mathbf{157.0} \text{ kN/m}$$

$$y = 10 \text{ mm}: \; p = \mathbf{198.0} \text{ kN/m}$$



Curve A approaches p_u = 205 kN/m; at 10 mm it is within 97% of it. At this depth the static and cyclic curves are **the same** (A = 0.9 in both).

Check the solution in GeoTerva ↗

Problem statement

EXAMPLE

Given

Pile	**pile, circular section, D = 0.60 m, L = 14 m, concrete: E I = 190 852 kN·m²**
Soil	**sand 0–4 m (φ′ = 32°) · clay 4–9 m (c_u = 40 kPa) · sand 9–17 m (φ′ = 32°)**
Unit weight, water	**γ = 18, γ_{sat} = 19 kN/m³; groundwater at 2.0 m**
Depth	**z = 0.6 m (1D), σ′ = 18(0.6) = 10.8 kPa; k = 20.7 MN/m³**
Sand 1	**C_1 = 2.281, C_2 = 2.947, C_3 = 36.81 (Example 3)**

REQUIRED

Find the static and cyclic **p–y** values for y = 10 mm.

Solution

SOLUTION

1 Ultimate resistance and A

$$p_{us} = (2.281(0.6) + 2.947(0.6))(10.8) = 33.9 \text{ kN/m}$$

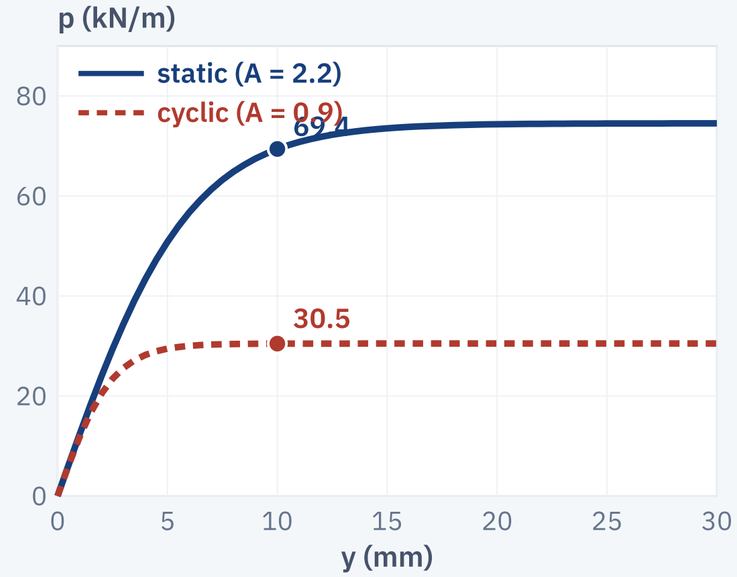
$$p_{ud} = 36.81(0.6)(10.8) = 238.6 \text{ kN/m} \Rightarrow p_u = \mathbf{33.9} \text{ kN/m}$$

$$A_{static} = \max(0.9;\ 3 - 0.8(1)) = 2.2 \qquad A_{cyclic} = 0.9$$

2 y = 10 mm: k z y = 20 700(0.6)(0.010) = 124.2

$$p_{static} = 2.2(33.9)\tanh\frac{124.2}{2.2(33.9)} = \mathbf{69.4} \text{ kN/m}$$

$$p_{cyclic} = 0.9(33.9)\tanh\frac{124.2}{0.9(33.9)} = \mathbf{30.5} \text{ kN/m}$$



In sand near the surface, cyclic loading reduces the resistance by **56%** (69.4 → 30.5 kN/m): it loses the A multiplier. This is why the head displacement increases under cyclic loading.

Check the solution in GeoTerva ↗

Matlock: soft clay, static and cyclic

THEORY

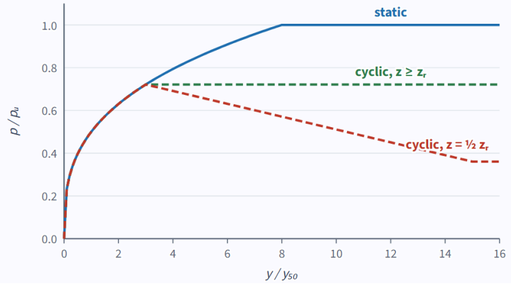
In clay, the module takes the **Matlock** (1970) soft clay curve in the form given by API RP 2A. The ultimate resistance is again the smaller of the near-surface (wedge) and deep (flow) values:

$$p_u = \min\left[\left(3 + \frac{\sigma'}{c_u} + J\frac{z}{D}\right) c_u D;\ 9c_u D\right]$$

$$p = \tfrac{1}{2} p_u \left(\frac{y}{y_{50}}\right)^{1/3} \quad (y \le 8y_{50}) \qquad p = p_u \ (y > 8y_{50}) \qquad y_{50} = 2.5\,\varepsilon_{50} D$$

ε_{50} and J with respect to c_u: below 48 kPa soft (0.020; 0.5), 48–96 medium (0.010; 0.25), above 96 stiff (0.005; 0.25).

Cyclic: after $3y_{50}$ the curve stays at 0.72 p_u ($z \ge z_R$); at shallower depths it **decreases linearly** to 0.72 p_u z/z_R at $15y_{50}$. z_R = max[6D / ((σ′/z)(D/c_u) + J); 2.5D].



static: p = ½ p_u (y / y_{50})^⅓ up to 8 y_{50}, then p = p_u

cyclic: the same up to 3 y_{50} (0.72 p_u there), then 0.72 p_u for z ≥ z_r, and nearer the surface falling linearly to 0.72 p_u z / z_r at 15 y_{50}

Problem statement

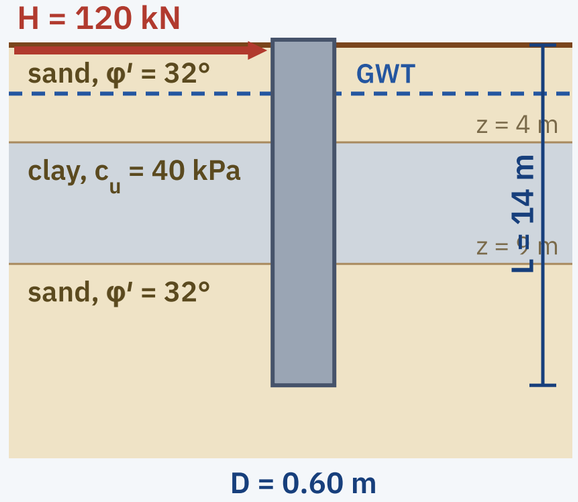
EXAMPLE

Given

Pile	**pile, circular section, D = 0.60 m, L = 14 m, concrete: E I = 190 852 kN·m²**
Soil	**sand 0–4 m (φ′ = 32°) · clay 4–9 m (c_u = 40 kPa) · sand 9–17 m (φ′ = 32°)**
Unit weight, water	**γ = 18, γ_{sat} = 19 kN/m³; groundwater at 2.0 m**
Depth	**z = 6.5 m (clay, 4–9 m); ε_{50} = 0.020, J = 0.5**
Layer	**c_u = 40 kPa**

REQUIRED

For y = 15 mm and 150 mm, find σ′, **p_u**, y_{50} and the **static** p–y values; find z_R and the plateau value of the **cyclic** curve.



Solution (1/2)

SOLUTION

1 σ′: sand 0–2 m moist (γ = 18), sand 2–4 m and clay 4–6.5 m submerged (γ_{sat} = 19)

$$\sigma' = 18(2) + (19 - 9.81)(2) + (19 - 9.81)(2.5) = \mathbf{77.4}\ \text{kPa}$$

2 Ultimate resistance and y_{50}

$$p_u = \min\left[\left(3 + \frac{77.4}{40} + 0.5\frac{6.5}{0.6}\right)(40)(0.6);\ 9(40)(0.6)\right]$$

$$= \min[248.4;\ 216.0] = \mathbf{216.0}\ \text{kN/m}$$

$$y_{50} = 2.5(0.020)(0.6) = \mathbf{30.0}\ \text{mm}$$

Solution (2/2)

SOLUTION

3 Static curve: $p = \frac{1}{2}\, p_u\, (y/y_{50})^{1/3}$

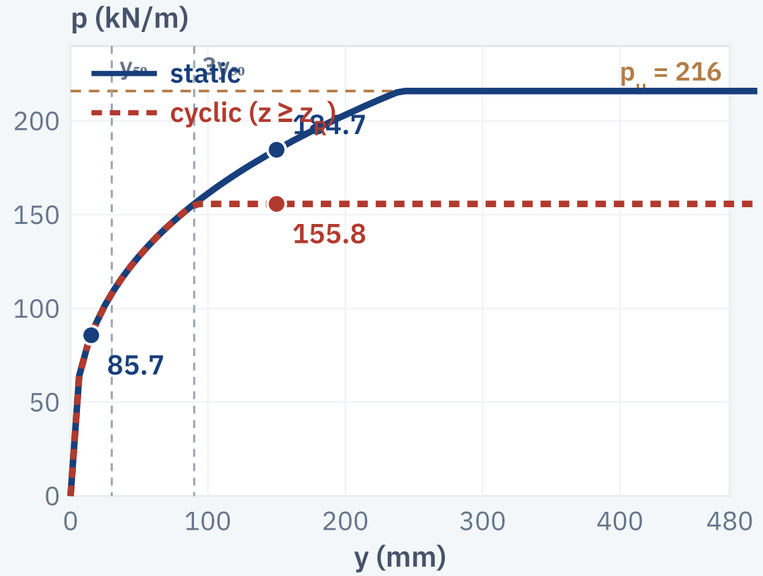
$$y = 15\ \text{mm}\ (0.5y_{50}):\ \ p = 0.5(216.0)(0.5)^{1/3} = \mathbf{85.7}\ \text{kN/m}$$

$$y = 150\ \text{mm}\ (5y_{50}):\ \ p = 0.5(216.0)(5)^{1/3} = \mathbf{184.7}\ \text{kN/m}$$

4 Cyclic curve: $z_R = \max[6(0.6)/((\sigma'/z)(D/c_u) + J);\ 2.5D] = \max[5.31;\ 1.5] = 5.31$ m; $z = 6.5 > z_R$

$$p_{perimeter}(y \geq 3y_{50}) = 0.72\, p_u = 0.7211(216.0) = \mathbf{155.8}\ \text{kN/m}$$

At y = 150 mm the static resistance is 184.7 and the cyclic resistance 155.8 kN/m: the cyclic load reduces the resistance in this clay by **16%**; up to $3y_{50}$ (90 mm) the two curves are identical.



Check the solution in GeoTerva ↗

Each curve corresponds to the soil at its own depth

THEORY

σ′ is the sum of the unit weights of the layers above (γ above the water table, $\gamma_{sat} - \gamma_w$ below); it is **not the average** of the layer, but increases with depth within the layer.

Layer z gives φ′ and k (sand) or c_u, ε_{50} and J (clay). The initial modulus k of the sand is taken relative to the water at the layer's location; in a layer intersected by water it is filled in according to the layer's mid-depth and can be edited manually.

The elements **never cross a layer boundary**: there is a node at every boundary; the stiffness of the soil jumps between elements, not within an element.

The calculation sheet gives, step by step, the curve values at the middle of each layer: σ′, coefficients, p_u, A or y_{50} — reproducible by hand (as in Examples 3 and 5 of this presentation).

The layers must extend to the pile tip; the soil below the tip is not defined and the tip is free. A soft clay over a stiff sand appears in the diagrams as a **break** at the boundary.

Finite elements and Newton

THEORY

The equation is solved by finite elements: the pile is divided into **cubic beam elements**, with y and slope θ at each node. The soil reaction is taken from the curve at **four Gauss points** in each element and integrated. The nonlinear equation K u + f(u) = F is solved by Newton's method:

$$(K + K_s(u_i))\,\Delta u = F - K u_i - f(u_i) \qquad K_s = \int N^T \frac{dp}{dy} N \, dz$$

dp/dy does not fall below **one-twentieth** of the secant stiffness: the matrix remains positive even where the curve is flat or descending. There is a step-reducing line search and, if it fails, a damped secant step.

Element length D/6 (3–15 cm); the result **does not** depend on the mesh down to the steps shown (element 0.4 → 0.025 m: head displacement 6.228 → 6.228 mm).

If the load exceeds what the soil can carry, **there is no equilibrium**; the module states this and gives no numbers. The solution is also not counted if the soil has displaced more than **five pile widths** below the ground surface.

The cyclic clay curve, or the response to axial loading, is not a single rising curve; it rises up to the **ultimate load**. The module reaches the load in steps from zero.

In each study two checks are written: Σ p Δz = H and moment at the tip = 0 (statics).

P–δ effect and buckling

THEORY

The axial load P acts along the **deflected shape** of the pile: it appears in the equation (the P d²y/dz² term) and in the geometric stiffness of each element; it reduces the lateral stiffness of the beam under compression and increases the displacement and moment. The module takes P as constant along the full length and the same at the head and the tip (in reality, shaft friction transfers the load to the soil; assuming it constant is on the safe side).

$$P_{cr} = \sqrt{k\,EI} \qquad (\text{free end, } k \text{ constant, long pile})$$

It is **half** the value 2√(kEI) for an infinite beam: a free end can rotate. A fixed head buckles at a higher load.

As P approaches the buckling load, the load–displacement rises without bound; the module states **"axial load exceeds the buckling load"**.

The combined resistance of the pile section to axial load and moment is a structural design check.

EXAMPLE PILE

For E I = 190,852 kN·m² and k = 9,000 kN/m² in elastic clay:

$$P_{cr} = \sqrt{9{,}000\,(190{,}852)} = 41{,}445 \text{ kN}$$

Problem statement

EXAMPLE

Given

Pile	**pile, circular section, D = 0.60 m, L = 14 m, concrete: E I = 190 852 kN·m²**
Soil	**sand 0–4 m (φ′ = 32°) · clay 4–9 m (c_u = 40 kPa) · sand 9–17 m (φ′ = 32°)**
Unit weight, water	**γ = 18, γ_{sat} = 19 kN/m³; groundwater at 2.0 m**
Loads	**H = 120 kN; axial load P = 800 kN (constant)**
Buckling	**k = k_h D = 9,000 kN/m² (elastic clay), free head**
Module	**P = 0: head 6.23 mm, M_{max} 151.8 kN·m**

REQUIRED

Is P = 800 kN **close** to the buckling load? How large is the P–δ effect?

Solution

SOLUTION

1 Buckling load (free head, k constant)

$$P_{cr} = \sqrt{kEI} = \sqrt{9{,}000(190{,}852)} = \mathbf{41{,}445}\ \text{kN} \qquad \frac{P}{P_{cr}} = \frac{800}{41{,}445} = \mathbf{0.019}$$

2 In the module with P = 800 kN (including P–δ)

$$y_{head}:\ 6.23 \rightarrow 6.32\ \text{mm}\ \ (+2\%) \qquad M_{max}:\ 151.8 \rightarrow 154.5\ \text{kN·m}\ \ (+2\%)$$

800 kN, only **2%** of the buckling load: the axial load does not bring the pile close to buckling; the P–δ effect increases the displacement by 2% and the moment by 2%. In this pile the axial load is secondary in the lateral analysis, but it cannot be neglected.

Check the solution in GeoTerva ↗

Free, fixed, and surcharge on the soil

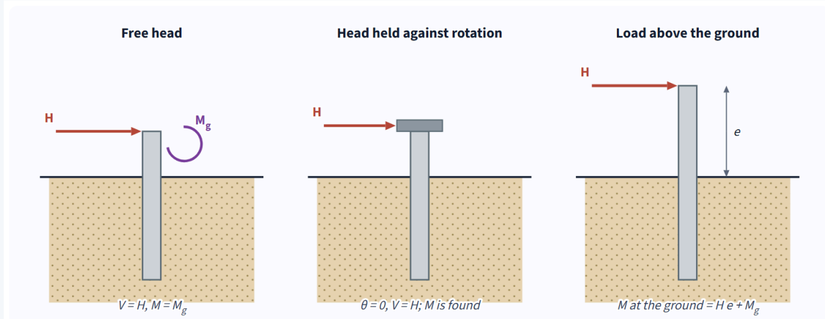
THEORY

Free head: at the load point V = H and M = M_g.

Fixed head: the slope is zero, the head moves laterally. On long, uniform, elastic ground it displaces **half as much** as the free-head case; the restraining moment M = −H/(2β) is greater than the maximum moment of the free pile. A real pile cap lies between the two cases.

Free length e: the pile above the ground bends like a cantilever. The shear force is H down to the ground, and the moment reaches **M(0) = M_g + H e** at the ground surface; there the soil takes over.

Head displacement = displacement at the ground surface + rotation at the surface × e + cantilever displacement of the free length (H e^3 / 3EI). The same load makes the head displacement with a free length **much larger** than that of the ground surface; the maximum moment is mostly in the ground or slightly below it.



Problem statement

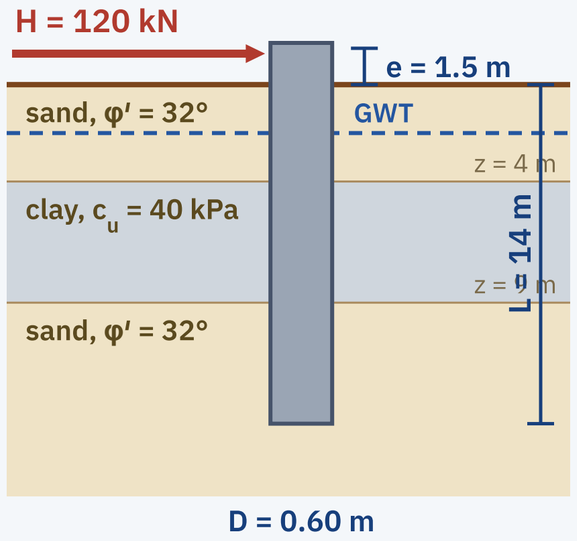
EXAMPLE

Given

Pile	**pile, circular section, D = 0.60 m, L = 14 m, concrete: E I = 190 852 kN·m²**
Soil	**sand 0–4 m (φ′ = 32°) · clay 4–9 m (c_u = 40 kPa) · sand 9–17 m (φ′ = 32°)**
Unit weight, water	**γ = 18, γ_{sat} = 19 kN/m³; groundwater at 2.0 m**
Free length	**load above the ground surface at e = 1.5 m; H = 120 kN, free head**
Modulus (at ground surface)	**y_{ground} = 11.12 mm, slope = 0.321° (toward the load)**

REQUIRED

Determine the **moment** and the **head displacement** at the ground surface; compare the result with 20.23 mm from the module.



Solution

SOLUTION

1 Moment at the ground surface (shear H over the free length, $M_g = 0$)

$$M(0) = M_g + H\,e = 0 + 120(1.5) = \mathbf{180}\ \mathrm{kN\cdot m}$$

2 Free-length cantilever displacement

$$\frac{H\,e^3}{3EI} = \frac{120(1.5)^3}{3(190{,}852)} = 0.71\ \mathrm{mm}$$

3 Head displacement = $y_{soil} + |\theta|\,e + H\,e^3/3EI$

$$y_{head} = 11.12 + 5.605(1.5) + 0.71 = \mathbf{20.23}\ \mathrm{mm}$$

By hand **20.23 mm**, in the module 20.23 mm: identical. With the load at ground level the head displaced 6.23 mm; 1.5 m of free length makes it **3.2 times** that. The maximum moment is 306 kN·m, 1.68 m below the ground surface (above the 180 kN·m at the ground surface).

Check the solution in GeoTerva ↗

Which number tells what?

THEORY

Head displacement y and rotation (slope dy/dz; negative for a pile lying in the load direction). If there is a free length, the value at the ground surface is also given.

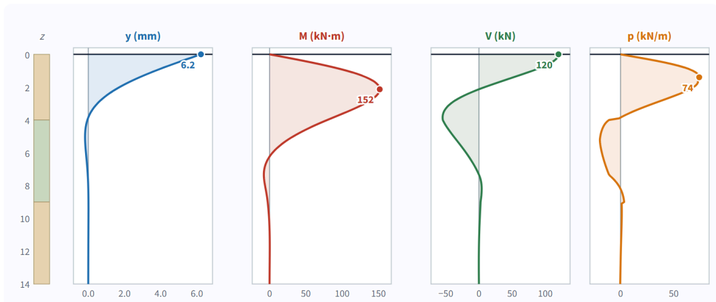
Maximum bending moment and its depth. For a restrained head, the head moment is also listed; it is not always the largest.

Depth of zero displacement: where the pile crosses its own axis; above it the soil moves toward the pile, below it away from the pile.

Shear force and soil reaction; the reaction and the end of the p–y curve are plotted side by side: it is seen how close the soil is to its limit.

Tip: at the tip of a sufficiently long pile there is no displacement. If the tip moves more than one-tenth of the ground surface displacement and more than 2 mm, the pile is **short** for this load: it rotates as a rigid body, and its length governs the result.

The p–y view plots the curves at five depths in the moving part and marks the **point reached** on each: the point at the flat start of the curve is elastic, the point near its end is at the limit.



worked pile: D = 0.6 m, L = 14 m, H = 120 kN at the ground; sand / clay / sand

Problem statement

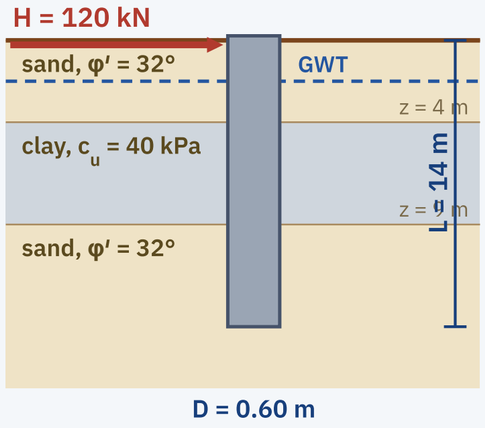
EXAMPLE

Given

Pile	pile, circular section, D = 0.60 m, L = 14 m, concrete: E I = 190 852 kN·m²
Soil	sand 0–4 m (φ′ = 32°) · clay 4–9 m (c_u = 40 kPa) · sand 9–17 m (φ′ = 32°)
Unit weight, water	γ = 18, γ_{sat} = 19 kN/m³; groundwater at 2.0 m
Module results	head y = 6.23 mm, slope 0.150°; M_{max} = 151.8 kN·m (z = 2.13 m); zero displacement at z = 3.85 m; tip 0.0019 mm
Maximum soil reaction	p = 74.0 kN/m, z = 1.40 m; A = max(0.9; 3 − 0.8(1.4)/0.6), p_u = 125.0 kN/m

REQUIRED

Is the pile **long** for this load? At what **percentage** of its limit is the soil working at a depth of 1.40 m? Interpret the static checks.



Solution (1/2)

SOLUTION

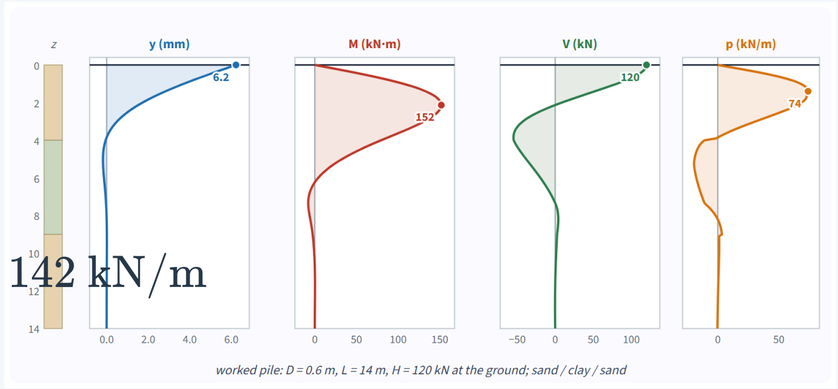
1 End condition: the pile-head movement is compared with 2 mm, one tenth of the displacement at the ground surface

$$y_{tip} = 0.0019\ \text{mm} \ll 0.1(6.23) = 0.62\ \text{mm} \ \Rightarrow\ \text{long pile}$$

2 At 1.40 m, how much of the soil is within its limit?

$$A = \max\left(0.9;\ 3 - \frac{0.8(1.4)}{0.6}\right) = 1.133\ ; \quad A\,p_u = 1.133(125.0) = 142\ \text{kN/m}$$

$$\frac{p}{Ap_u} = \frac{74.0}{142} = \mathbf{0.52}$$



worked pile: D = 0.6 m, L = 14 m, H = 120 kN at the ground; sand / clay / sand

Solution (2/2)

SOLUTION

3 Static checks

$$\sum p\,\Delta z = 120.000\ \text{kN} = H \qquad M_{tip}\ (\text{from static}) = 0.000\ \text{kN}\cdot\text{m} \qquad 7\ \text{iteration}$$

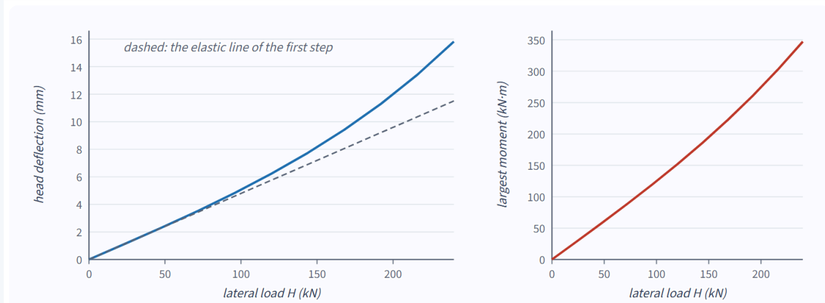
The pile is **long**; at 1.40 m the soil works at **52%** of its limit (not at the end of the curve). Both checks also hold: these are checks of the iteration and the bookkeeping, **not of the model's accuracy**.

Check the solution in GeoTerva ↗

Load corresponding to the allowable value

THEORY

Since the soil is nonlinear, displacement is not proportional to load; the **load at the allowable displacement** cannot be read from the result at the working load. The module increases the load and moment proportionally from the start (a factor λ on H and M_g, with the axial load held constant), re-solves the equation at each step, and finds the λ at which the head displacement reaches the allowable value (or the maximum moment reaches the section capacity): $\mathbf{H_y}$ or $\mathbf{H_M}$.



If the soil cannot carry the allowable value (if equilibrium is not reached before it), the module reports **"not reached"** and the curve ends at the last load solved. For cyclic clay or axial load the curve bends: this is the **limit load of the soil**.

This is not the capacity of the pile: in the module the pile is elastic; a pile yielding in bending may fail before the soil does. The entered moment capacity is the check for this.

The module does not provide a factor of safety and does not take one from a code: the allowable displacement and moment capacity are the engineer's responsibility.

Problem statement

EXAMPLE

Given

Pile	**pile, circular section, D = 0.60 m, L = 14 m, concrete: E I = 190 852 kN·m²**
Soil	**sand 0–4 m (φ′ = 32°) · clay 4–9 m (c_u = 40 kPa) · sand 9–17 m (φ′ = 32°)**
Unit weight, water	**γ = 18, γ_{sat} = 19 kN/m³; groundwater at 2.0 m**
Nonlinear solution	**For H = 30, 60, …, 270 kN the head displacement: 1.44 · 2.93 · 4.51 · 6.23 · 8.15 · 10.33 · 12.86 · 15.82 · 19.29 mm**
Allowable	**y_{all} = 10 mm; moment capacity M_{cap} = 200 kN·m**

REQUIRED

H_y, the load at y_{all} = 10 mm; H_M, the load corresponding to M_{cap}; what would the error be if linear proportionality were used?

Solution

SOLUTION

1 Linear proportion (from 6.23 mm at the working load) and the exact solution

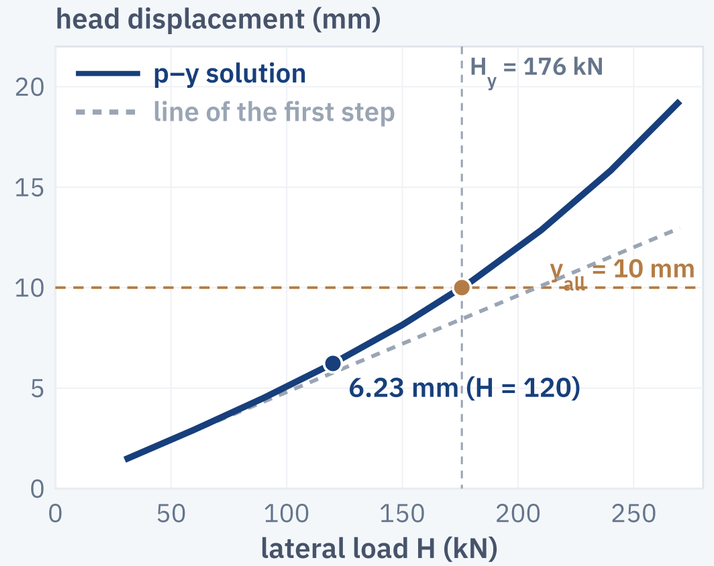
$$H_{y,linear} = 120\,\frac{10}{6.23} = 192.7\ \text{kN}$$

$$H_y = \mathbf{175.7}\ \text{kN}\ (\lambda = 1.464)$$

2 Load corresponding to the moment capacity

$$H_M = \mathbf{153.6}\ \text{kN}\ \ (\lambda = 1.280;\ y_{head} = 8.39\ \text{mm})$$

A linear proportionality would give the load **10% too high** (193 ≠ 176 kN): as the soil softens, displacement grows faster than the load. The **governing** quantity is the moment: H_M = 154 kN, at which the head displacement is only 8.4 mm; at the load corresponding to the allowable displacement (H_y = 176 kN) the moment would be 234 kN·m.



Check the solution in GeoTerva ↗

Problem statement

EXAMPLE

Given

Pile	**pile, circular section, D = 0.60 m, L = 14 m, concrete: E I = 190 852 kN·m²**
Soil	**sand 0–4 m (φ′ = 32°) · clay 4–9 m (c_u = 40 kPa) · sand 9–17 m (φ′ = 32°)**
Unit weight, water	**γ = 18, γ_{sat} = 19 kN/m³; groundwater at 2.0 m**
To be compared	**head condition · free length · cyclic load · axial load (800 kN) · cracked section (E I × 0.6) · elastic soil (n_h = 8 MN/m³, k_h = 15 MN/m³)**

REQUIRED

In each case compare the head displacement and the maximum moment with respect to the working load.

Solution

SOLUTION

Condition	y_{head} (mm)	M_{max} (kN·m)	y / y_0	M / M_0
As entered: free, static, at ground level	6.23	152	1.00	1.00
Fixed against rotation at the top (M_{top})	2.24	-175	0.36	1.15
The load is 1.5 m above the ground	20.23	306	3.25	2.01
Cyclic loading (API)	7.28	172	1.17	1.13
Axial load P = 800 kN	6.32	155	1.02	1.02
Cracked section, E I × 0.6	7.93	140	1.27	0.92
Elastic soil (n_h, k_h)	10.49	173	1.68	1.14

The head condition reduces the displacement by a factor of 0.36 and increases the moment; **free length** has the greatest effect (×3.2 displacement, ×2.0 moment). **Cyclic loading** increases the displacement by 17%, **a cracked section** by 27%, but reduces the moment (the pile is more flexible). The elastic soil is a separate matter: it is not a check of the p–y curves.

Check the solution in GeoTerva ↗

The assumptions behind the numbers

THEORY

Single pile: a pile standing alone, with no neighbors, by its p–y curves. A laterally loaded group displaces more than a single pile at the same pile load; the group is the subject of the pile group module with p-multipliers for the rows.

Pile: single section over the full length, linear elastic; E I constant. A cracked concrete section is represented by the E I multiplier and by the moment M_{cap} at which the section yields.

Soil: independent springs, no shear between them; the curves come from tests performed on specific soils: sand depends on φ', clay on c_u and ε_{50}. It describes sand and soft-to-medium clay; stiff clay, weathered rock, cemented soil and liquefiable sand are described by their own curves.

Loading: static or cyclic with API reduction; applied first. **Axial load** is constant over the full length.

Numerical solution: finite elements and nonlinear iteration are exact for the given equation down to the steps shown; the approximation lies in the equation (the soil is represented by springs). It agreed within 0.01% for 14 piles with two independent programs (finite differences, in another language).

The p–y method is fundamental to the design of laterally loaded piles in practice; the values should be compared with the project's allowable values and, if available, with the **load test**.

Summary

Key points

The pile is a **beam on non-linear springs**: $EI\,y'''' + P\,y'' + p(y, z) = 0$; the soil law is the p–y curve

Elastic: sand $n_h z \rightarrow T = (EI/n_h)^{1/5}$; clay $k \rightarrow \beta = (k/4EI)^{1/4}$, $y_0 = 2H\beta/k$, $M_{max} = 0.3224\ H/\beta$

Sand: $p = A\,p_u \tanh(kzy/Ap_u)$, $p_u = \min[(C_1 z + C_2 D)\sigma'; C_3 D\sigma']$ · **Clay**: $p = \frac{1}{2} p_u (y/y_{50})^{1/3}$, $p_u = \min[(3 + \sigma'/c_u + Jz/D)c_u D; 9c_u D]$

Cyclic loading reduces near-surface resistance: $A = 0.9$ in sand, $0.72p_u$ in clay beyond $3y_{50}$

The **head condition**, **free length**, **axial load** and **cracked section** change the result markedly; the free length most of all

Displacement is not proportional to load: the **load at the allowable displacement** is found by re-solving (H_y, H_M)

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NOTE

The figures in this presentation are taken from the GeoTerva Theory page; all numerical examples are produced by the module's own calculation engine, and the module gives the same values.

GeoTerva

Online geotechnical analysis

From theory to practice

Calculate the example pile in this presentation step by step in GeoTerva: the numbers are the same.

Open the module →

Theory page →

Open on your phone
The example modules need no registration.

If you would like to use them in your lectures: info@geoterva.com