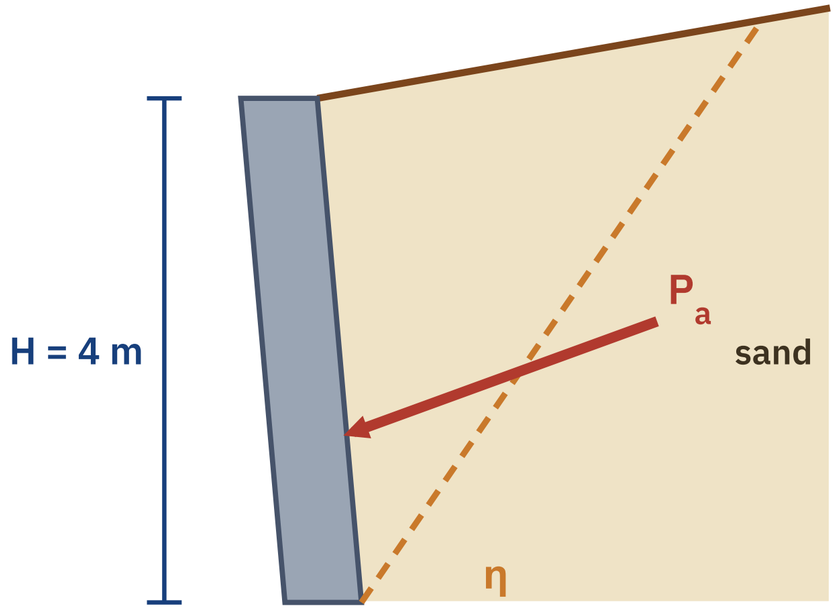
Geotechnical lecture presentation · Lateral earth pressure

# Coulomb lateral earth pressure

Lecture: Coulomb active and passive earth pressure for a planar slip wedge behind the wall, wall friction, inclined wall face and sloping fill; surcharge, groundwater and layered fill; a worked example for each topic.

Trial wedge | Wall friction δ′ | Active and passive | Surcharge · water · layer | Cohesive layer



# What will we learn in this lecture?

- Coulomb's **trial wedge** idea: equilibrium of the three forces (W, F, $P_a$) and the most critical wedge
- How wall friction (δ′), wall face inclination (θ) and backfill slope (α) change the **$K_a$ and $K_p$** coefficients
- That the **active** force is the **largest** of the wedges and the **passive** force is the **smallest** of the wedges
- To find the location of the force (H/3), its direction (δ′ from the face normal) and its horizontal – vertical components
- **Surcharge** ($q_e = q \cos\theta / \cos(\theta - \alpha)$), **groundwater** and layered fill
- When the Coulomb solution is **unsafe** in the passive case

LECTURE FLOW

Each topic is covered in **three steps**:

1 · **Theory**: what is the rule, why is it so?

2 · **Problem statement**: given and required

3 · **Solution**: step by step, with numbers

Forces are per unit length of wall (kN/m); angles in degrees; θ is the angle of the wall face with the vertical, α is the angle of the backfill with the horizontal.

# Coulomb's planar wedge

THEORY

Coulomb (1776) treats the soil behind the wall as a **rigid wedge** sliding on a plane passing through the heel and, unlike Rankine, accounts for the **friction** between the wall and the soil. In cohesionless fill, three forces act on the trial wedge:

**W**: weight of the wedge

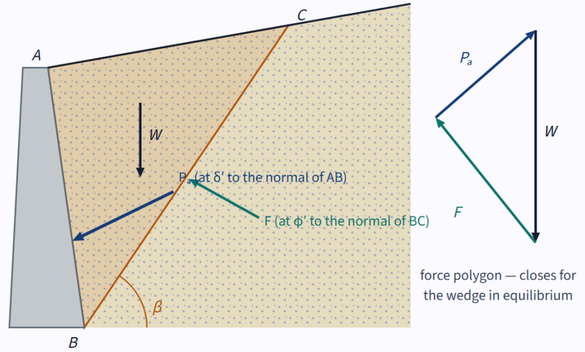
**F**: the resultant on slip plane BC, inclined at φ′ from the normal to the plane

**$P_a$**: wall force, inclined at δ′ from the normal to the wall face

If the plane makes an angle β with the horizontal, the sine rule in the force polygon:

$$\frac{W}{\sin(90^\circ + \theta + \delta' - \beta + \varphi')} = \frac{P_a}{\sin(\beta - \varphi')}$$

$$P_a = \tfrac{1}{2}\gamma H^2 \, \frac{\cos(\theta - \beta)\cos(\theta - \alpha)\sin(\beta - \varphi')}{\cos^2\theta \, \sin(\beta - \alpha) \, \sin(90^\circ + \theta + \delta' - \beta + \varphi')}$$



# Problem statement

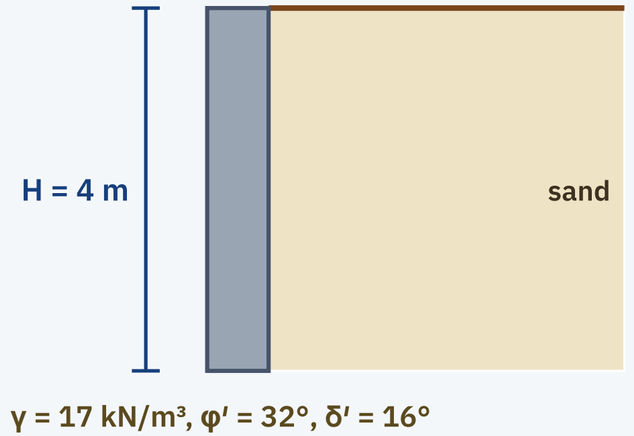
EXAMPLE

## Given

| | |
|---|---|
| Wall | $H = 4$ **m, vertical ($\theta = 0$); fill surface horizontal ($\alpha = 0$)** |
| Soil | **dry sand:** $\gamma = 17\ \mathrm{kN/m^3}$, $\varphi' = 32^\circ$**; wall friction** $\delta' = 16^\circ = \varphi'/2$ |

## REQUIRED

For trial planes β = 45°, 60° and 75°, **$P_a(\beta)$**; the **largest** force and its angle β.



# Solution

SOLUTION

**1** Vertical wall, horizontal fill: θ = α = 0 ⇒ $P_a(\beta) = \tfrac{1}{2}\gamma H^2 \cdot \cos\beta \sin(\beta - \varphi') / [\sin\beta \cos(\beta - \varphi' - \delta')]$

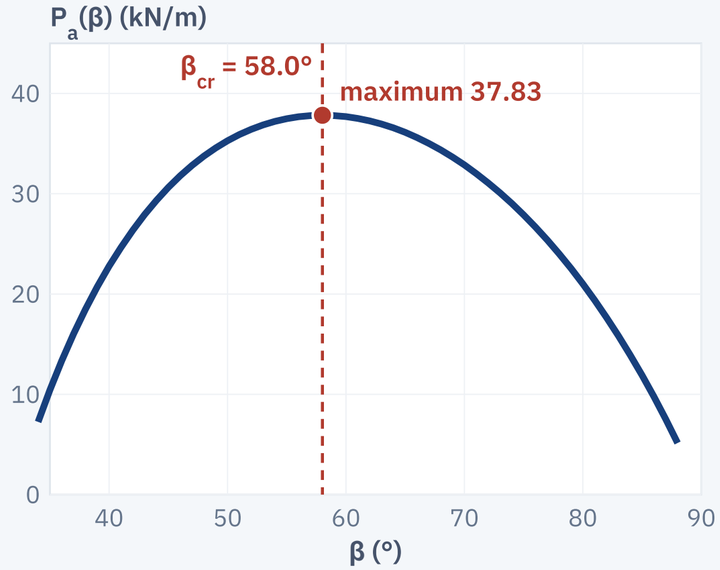
$$\beta = 45^\circ: \; P_a = \mathbf{30.64}\ \mathrm{kN/m} \qquad \beta = 60^\circ: \; P_a = \mathbf{37.69}\ \mathrm{kN/m}$$

$$\beta = 75^\circ: \; P_a = \mathbf{27.89}\ \mathrm{kN/m}$$

**2** Largest trial wedge ($dP_a/d\beta = 0$)

$$\beta_{cr} = \mathbf{58.0}^\circ \qquad P_{a,max} = \mathbf{37.83}\ \mathrm{kN/m}$$

**The largest of the trial wedges, 37.83 kN/m** (β = 58.0°), is Coulomb's active force; steeper or flatter wedges require a smaller force. The wall must be able to retain only the most critical wedge.




Check the solution in GeoTerva ↗

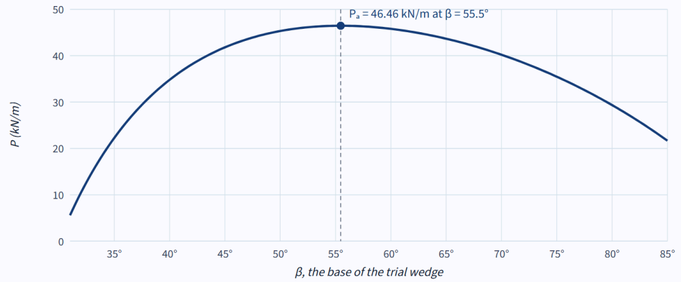
# Coulomb active coefficient $K_a$

THEORY

Only β was unknown. The most critical wedge that the wall must retain is the maximum of $P_a(\beta)$ ($dP_a/d\beta = 0$); its solution:

$$K_a = \frac{\cos^2(\varphi' - \theta)}{\cos^2\theta\cos(\delta' + \theta)\left[1 + \sqrt{\dfrac{\sin(\delta' + \varphi')\sin(\varphi' - \alpha)}{\cos(\delta' + \theta)\cos(\theta - \alpha)}}\right]^2} \qquad P_a = \tfrac{1}{2}K_a\gamma H^2$$

| φ′ | δ′ = 0° | δ′ = 5° | δ′ = 10° | δ′ = 15° | δ′ = 20° | δ′ = 25° |
|---|---|---|---|---|---|---|
| 28° | 0.3610 | 0.3448 | 0.3330 | 0.3251 | 0.3203 | 0.3186 |
| 32° | 0.3073 | 0.2945 | 0.2852 | 0.2791 | 0.2755 | 0.2745 |
| 36° | 0.2596 | 0.2497 | 0.2426 | 0.2379 | 0.2354 | 0.2350 |
| 40° | 0.2174 | 0.2098 | 0.2045 | 0.2011 | 0.1994 | 0.1995 |




The force acts at **H/3** above the base, inclined at **δ′** from the normal to the wall face; it makes an angle θ + δ′ with the horizontal.

Rankine for α = θ = δ′ = 0. Wall friction **reduces** the active force.

# Problem statement

EXAMPLE

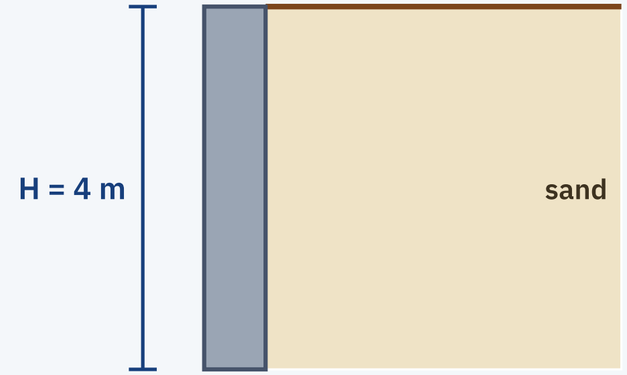
**REQUIRED**

**$K_a$, $P_a$,** height of action, horizontal and vertical components; comparison with Rankine.

**Given**

Wall and soil

**The wall in Example 1: $H = 4$ m, $\theta = \alpha = 0$ ; $\gamma = 17\ \mathrm{kN/m^3}$, $\varphi' = 32°$, $\delta' = 16°$**



# Solution

SOLUTION

**1** Coefficient (θ = α = 0)

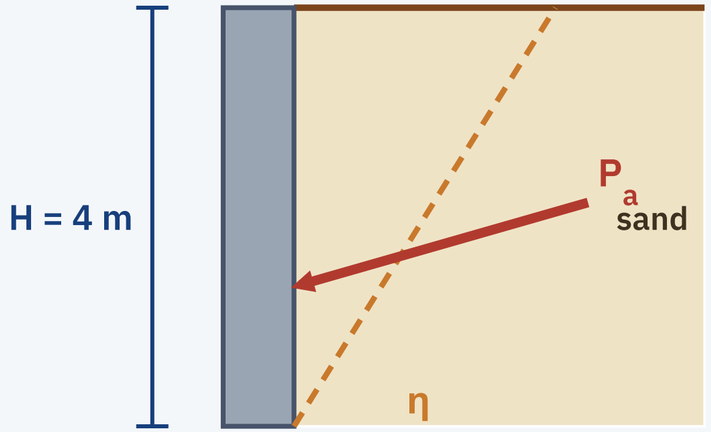
$$\sqrt{\frac{\sin 48^\circ \sin 32^\circ}{\cos 16^\circ}} = 0.6401$$

$$K_a = \frac{\cos^2 32^\circ}{\cos 16^\circ \,(1 + 0.6401)^2} = \frac{0.7192}{2.5856} = \mathbf{0.2782}$$

**2** Force, location and components (δ′ = 16° from the horizontal)

$$P_a = \tfrac{1}{2}(0.2782)(17)(4)^2 = \mathbf{37.83}\ \text{kN/m} \qquad \bar{z} = \tfrac{4}{3} = 1.33\ \text{m}$$

$$P_h = P_a \cos 16^\circ = 36.36 \qquad P_v = P_a \sin 16^\circ = 10.43\ \text{kN/m}$$




$P_a$ = **37.83 kN/m**; the same as with the largest of the trial wedges (Example 1). Rankine (δ′ = 0, $K_a$ = 0.3073) would give 41.79 kN/m: wall friction reduced the force by 9%.

Check the solution in GeoTerva ↗

# Problem statement

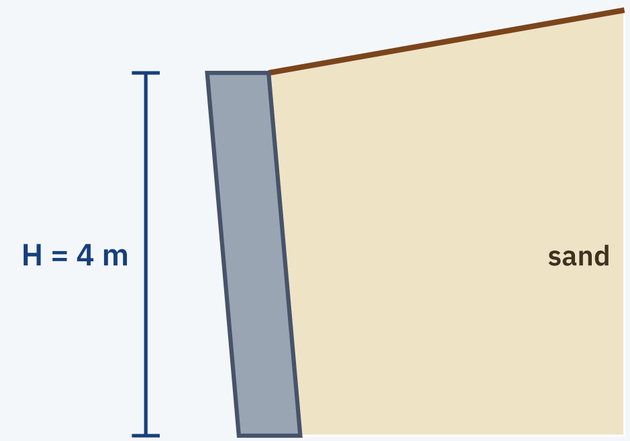
EXAMPLE

## Given

| | |
|---|---|
| Wall | $H = 4$ **m; face inclined at** $\theta = 5^\circ$ **from vertical** |
| Fill | **surface slope** $\alpha = 10^\circ$**;** $\gamma = 15\ \mathrm{kN/m^3},\ \varphi' = 30^\circ$ |
| Wall friction | $\delta' = 15^\circ$ **(**$\delta'/\varphi' = \frac{1}{2}$**)** |

## REQUIRED

**$K_a$**, **$P_a$**, its direction and location; critical wedge β. The module opens with this wall.



# Solution (1/2)

SOLUTION

**1** Numerator and root term

$$\cos^2(30^\circ - 5^\circ) = 0.8214 \qquad \cos^2 5^\circ \cos(15^\circ + 5^\circ) = 0.9326$$

$$\sqrt{\frac{\sin 45^\circ \sin 20^\circ}{\cos 20^\circ \cos(-5^\circ)}} = \sqrt{0.2583} = 0.5083$$

**2** Coefficient

$$K_a = \frac{0.8214}{0.9326\,(1 + 0.5083)^2} = \mathbf{0.3872}$$

# Solution (2/2)

SOLUTION

**3** Force and location

$$P_a = \tfrac{1}{2}\gamma H^2 K_a = \tfrac{1}{2}(15)(4)^2(0.3872) = \mathbf{46.46}\ \mathrm{kN/m}$$

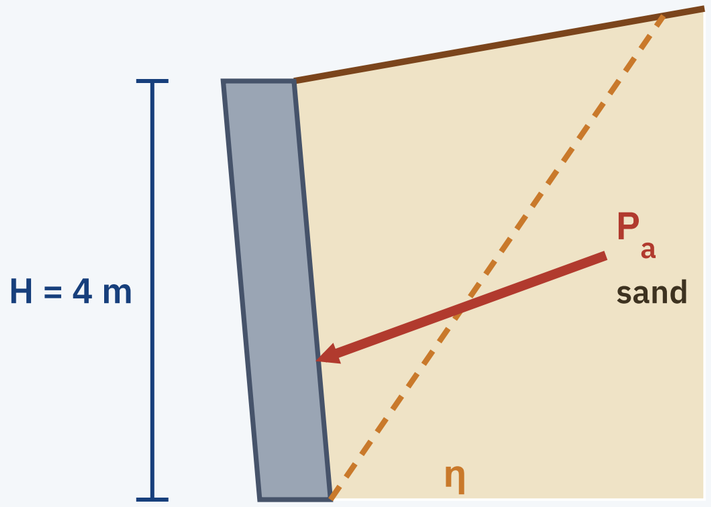
$$\bar{z} = \tfrac{H}{3} = \mathbf{1.33}\ \mathrm{m}$$

**4** Direction: δ′ = 15° from the normal, θ + δ′ = 20° from the horizontal

$$P_h = P_a \cos 20^\circ = 43.66\ \mathrm{kN/m}$$

$$P_v = P_a \sin 20^\circ = 15.89\ \mathrm{kN/m}$$

$P_a$ = **46.46 kN/m**, 1.33 m above the base. The critical wedge makes **β ≈ 55.5°** with the horizontal (apex of the trial wedges).




Check the solution in GeoTerva ↗

# Problem statement

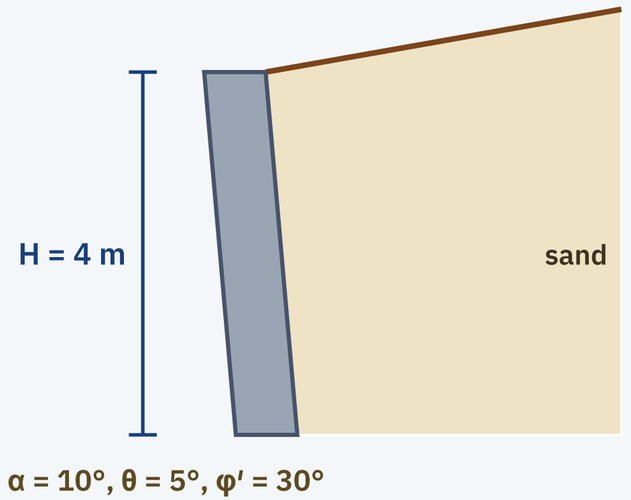
EXAMPLE

## Given

| | |
|---|---|
| Wall and fill | **The wall in Example 3: $\theta = 5^\circ,\ \alpha = 10^\circ,\ \varphi' = 30^\circ$; for comparison, vertical wall, horizontal fill ($\theta = \alpha = 0$)** |
| Variable | $\delta' = 0^\circ,\ 5^\circ,\ 10^\circ,\ 15^\circ,\ 20^\circ,\ 25^\circ,\ 30^\circ$ |

**REQUIRED**

For each δ′, **$K_a$**; at which δ′ is $K_a$ **smallest**? The recommended δ′/φ′ range.



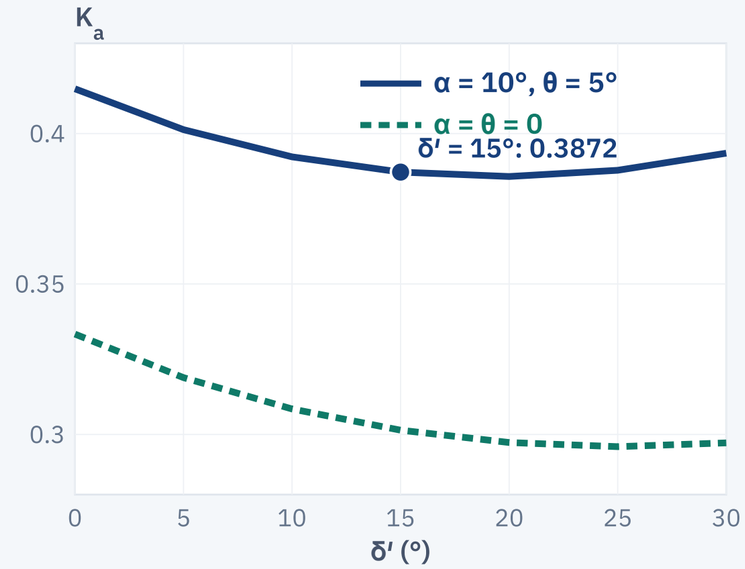
# Solution

SOLUTION

**1** Seven δ′ values with the same formula

| δ′ | 0° | 5° | 10° | 15° | 20° | 25° | 30° |
|---|---|---|---|---|---|---|---|
| $K_a$ (θ = 5°, α = 10°) | 0.4149 | 0.4013 | 0.3923 | 0.3872 | 0.3857 | 0.3878 | 0.3935 |
| $K_a$ (θ = α = 0) | 0.3333 | 0.3189 | 0.3085 | 0.3014 | 0.2973 | 0.2959 | 0.2972 |

As δ′ increases, $K_a$ first **decreases** (from 0.415 at δ′ = 0 to 0.387 at 15°), reaches a minimum at δ′ ≈ 20–25°, and then **increases** again. In practice δ′ = (1/2 – 2/3)φ′ is adopted; taking δ′ = φ′ is not conservative, since it depends on the wall sliding downward relative to the soil.




Check the solution in GeoTerva ↗

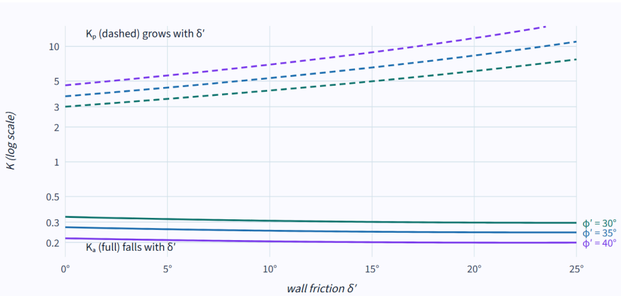
# Coulomb passive coefficient $K_p$

THEORY

When the wall is pushed into the soil, the resultant and the wall force normal turn to the **other side**; the passive force is the **smallest** of the trial wedges:

$$K_p = \frac{\cos^2(\varphi' + \theta)}{\cos^2\theta\cos(\delta' - \theta)\left[1 - \sqrt{\dfrac{\sin(\varphi' + \delta')\sin(\varphi' + \alpha)}{\cos(\delta' - \theta)\cos(\alpha - \theta)}}\right]^2} \qquad P_p = \tfrac{1}{2}K_p\gamma H^2$$

| φ′ | δ′ = 0° | δ′ = 5° | δ′ = 10° | δ′ = 15° | δ′ = 20° |
|---|---|---|---|---|---|
| 15° | 1.698 | 1.901 | 2.131 | 2.403 | 2.735 |
| 20° | 2.040 | 2.313 | 2.635 | 3.029 | 3.525 |
| 25° | 2.464 | 2.833 | 3.285 | 3.855 | 4.597 |
| 30° | 3.000 | 3.505 | 4.143 | 4.977 | 6.105 |
| 35° | 3.690 | 4.391 | 5.309 | 6.555 | 8.324 |
| 40° | 4.599 | 5.593 | 6.946 | 8.872 | 11.771 |




Wall friction **increases the passive force rapidly** (φ′ = 30°: $K_p$ rises from 3.00 to 6.11 for δ′ = 0 → 20°).

For large δ′, the planar wedge gives **more than** the passive resistance (unsafe): logarithmic spiral module.

# Problem statement

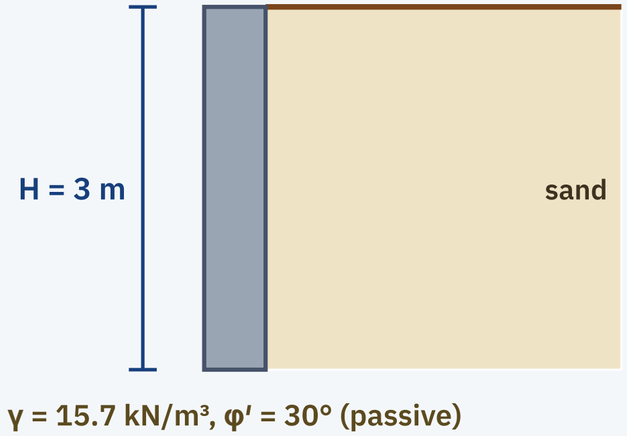
EXAMPLE

**REQUIRED**

For each δ′: **$K_p$**, **$P_p$** and their components; for δ′ = 15°, comparison with curved-surface solutions (302–326 kN/m).

**Given**

| | |
|---|---|
| Wall | $H = 3$ **m, vertical; horizontal fill** |
| Soil | **dry sand:** $\gamma = 15.7\ \mathrm{kN/m^3},\ \varphi' = 30^\circ$ |
| Wall friction | $\delta' = 0^\circ,\ 10^\circ,\ 15^\circ,\ 20^\circ$ |



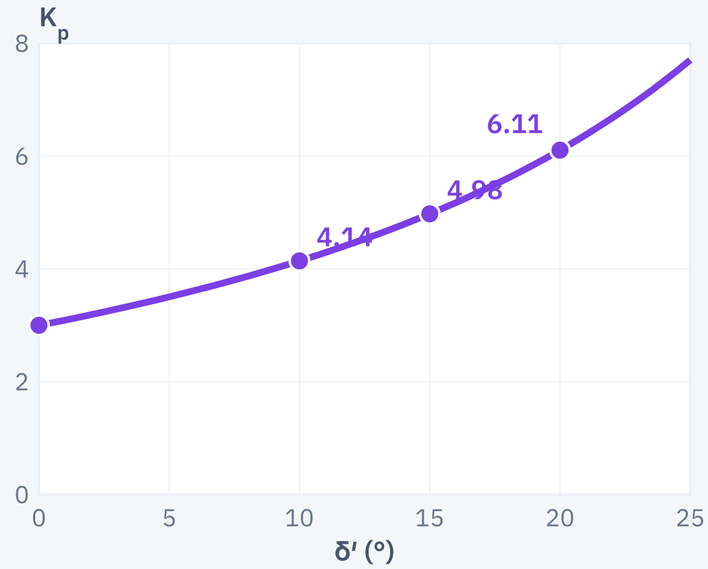
# Solution

SOLUTION

**1** $K_p$ formula, $P_p = \frac{1}{2}K_p\gamma H^2 = \frac{1}{2}K_p(15.7)(9) = 70.65\ K_p$

| δ′ | $K_p$ | $P_p$ (kN/m) | $P_h$ | $P_v$ | β (critical wedge) |
|---|---|---|---|---|---|
| 0° | 3.000 | 212.0 | 212.0 | 0.0 | 30.0° |
| 10° | 4.143 | 292.7 | 288.3 | 50.8 | 23.4° |
| 15° | 4.977 | 351.6 | 339.6 | 91.0 | 20.7° |
| 20° | 6.105 | 431.3 | 405.3 | 147.5 | 18.1° |

from δ′ = 0 to 20°, $P_p$ goes from 212.0 to 431.3 kN/m: it rises to **2.0 times**. At δ′ = 15° Coulomb gives 351.6 kN/m; curved-surface solutions lie in the range 302–326 kN/m, i.e. the planar wedge passive overestimates by about **8 %** (unsafe).




Check the solution in GeoTerva ↗

# Module extension of the Coulomb solution

THEORY

**Surcharge q** (vertical, per unit area of the inclined surface) is added to the weight of each trial wedge in the same proportion as the soil; the critical wedge and $K_a$ do not change. The equivalent stress per vertical unit depth:

$$q_e = q\,\frac{\cos\theta}{\cos(\theta-\alpha)} \qquad P_a = \tfrac{1}{2}K_a\gamma H^2 + K_a q_e H$$

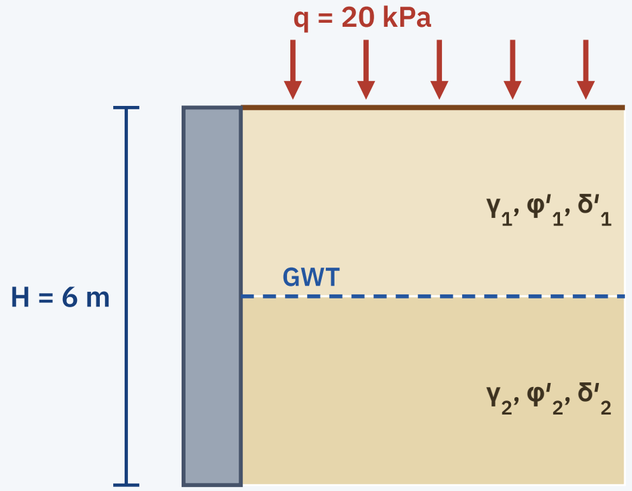
The first part is H/3 above the base, the second part H/2. **In layered fill** (boundaries horizontal, thicknesses measured vertically from the top A of the wall face) each layer takes its own $K_i$ coefficient with its own γ, φ′ and δ′:

$$\sigma'_v = q_e + \sum \gamma_i h_i \qquad \sigma' = K_i\,\sigma'_v\cos\theta$$

σ′ acts on unit length of the wall face, inclined from the normal at the layer's δ′ angle; K jumps at a boundary where it changes.

**Groundwater** (horizontal, $z_w$) below: $\gamma' = \gamma_{sat} - \gamma_w$; $u = \gamma_w(z - z_w)$ is added **perpendicular** to the wall. Forces are resolved into components normal (N) and parallel (T) to the face and summed: $P = \sqrt{(N^2 + T^2)}$, $\beta = \tan^{-1}(T/N)$.

## LAYERED FILL AND WATER



# Problem statement

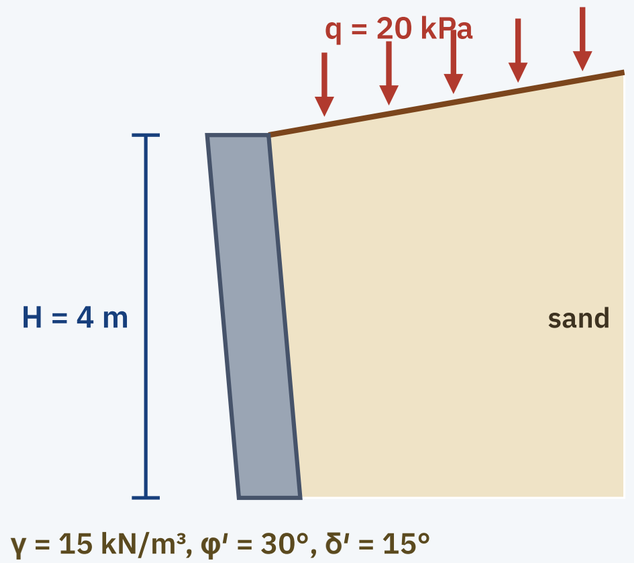
EXAMPLE

## Given

| | |
|---|---|
| Wall and soil | **The wall in Example 3:** $H = 4$ **m,** $\theta = 5^\circ,\ \alpha = 10^\circ$**;** $\gamma = 15\ \mathrm{kN/m^3},\ \varphi' = 30^\circ,\ \delta' = 15^\circ$ |
| Surcharge | **uniformly distributed** $q = 20$ **kPa on inclined surface (vertical to the surface)** |

## REQUIRED

$q_e$, total $P_a$ and the height of its action; the share of the surcharge.



# Solution

SOLUTION

**1** Equivalent vertical stress

$$q_e = q\frac{\cos\theta}{\cos(\theta-\alpha)} = 20\frac{\cos 5^\circ}{\cos(-5^\circ)} = \mathbf{20.00}\ \text{kPa}$$

**2** Soil and surcharge forces ($K_a$ = 0.3872)

$$P_\gamma = \tfrac{1}{2}K_a\gamma H^2 = 46.46\ \text{kN/m}\ \ (H/3 = 1.33\ \text{m})$$

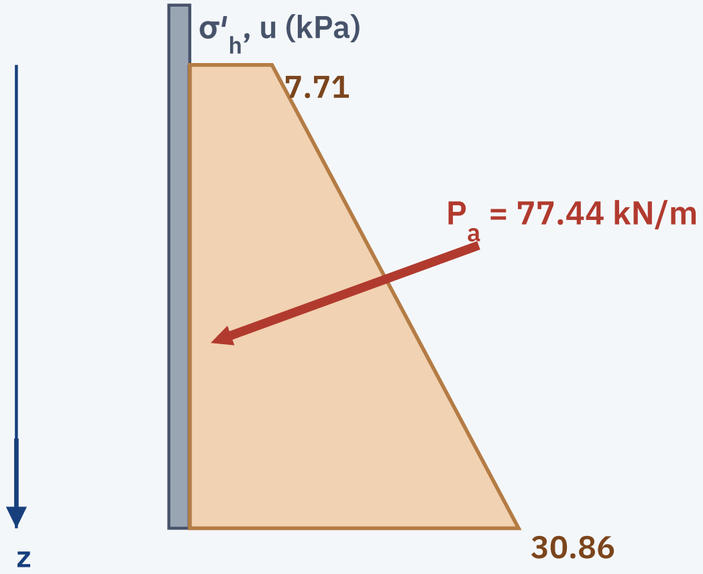
$$P_q = K_a q_e H = 0.3872(20)(4) = 30.97\ \text{kN/m}\ \ (H/2 = 2.00\ \text{m})$$

**3** Total and location

$$P_a = 46.46 + 30.97 = \mathbf{77.44}\ \text{kN/m}$$

$$\bar{z} = \frac{46.46(1.33) + 30.97(2.00)}{77.44} = \mathbf{1.60}\ \text{m}$$

The surcharge increased the force by 67% ($P_a$ = 77.44 kN/m); the point of application rose from 1.33 m to **1.60 m**.




Check the solution in GeoTerva ↗

# Problem statement

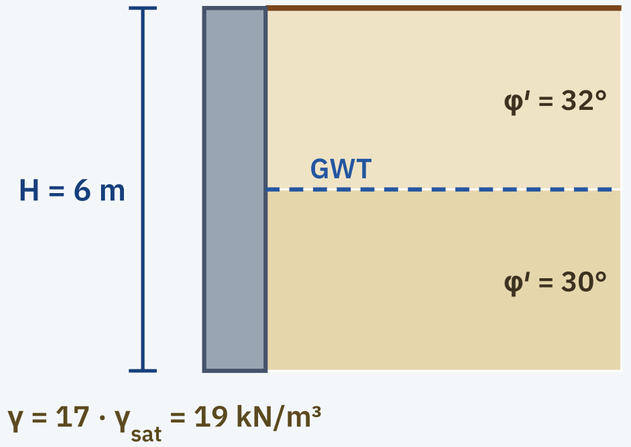
EXAMPLE

## Given

| | |
|---|---|
| Wall | $H = 6$ **m, vertical; horizontal fill; no surcharge** |
| Layer 1 (0–3 m) | $\gamma = 17\ \mathrm{kN/m^3},\ \varphi' = 32^\circ,\ \delta' = 16^\circ$ |
| Layer 2 (3–6 m) | $\gamma_{sat} = 19\ \mathrm{kN/m^3},\ \varphi' = 30^\circ,\ \delta' = 15^\circ$ |
| Groundwater level | **At 3 m depth** |

## REQUIRED

Pressures in each layer; soil and water forces; **P**, β and height of action.



# Solution (1/2)

SOLUTION

**1** Coefficients

$$K_1(32^\circ, 16^\circ) = \mathbf{0.2782} \qquad K_2(30^\circ, 15^\circ) = \mathbf{0.3014}$$

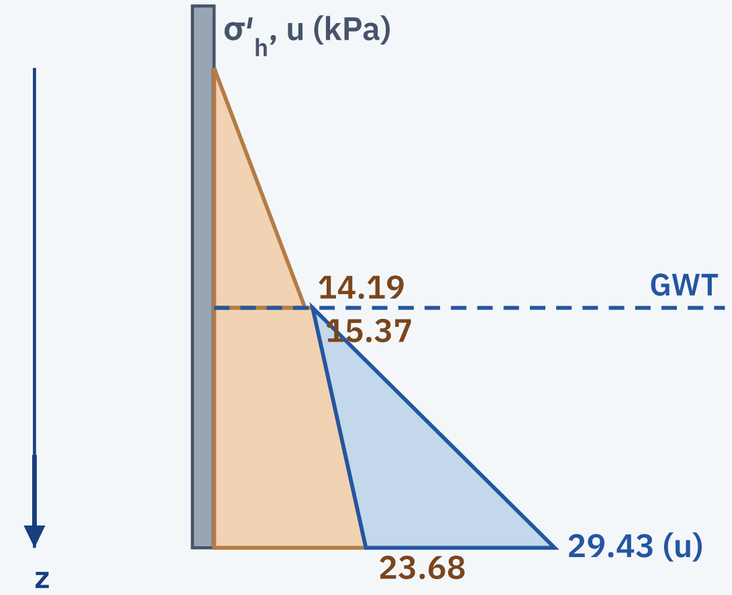
**2** Stresses: $\sigma'_v(3) = 17(3) = 51$; $\sigma'_v(6) = 51 + 3(19 - 9.81) = 78.57$ kPa

$$p_1 = 0.2782(51) = \mathbf{14.19}\ \text{kPa (top)}$$

$$p_2 = 0.3014(51) = \mathbf{15.37}\ \text{kPa (bottom)}$$

$$p(6) = 0.3014(78.57) = \mathbf{23.68}\ \text{kPa}$$

$$u(6) = 9.81(3) = \mathbf{29.43}\ \text{kPa}$$



# Solution (2/2)

SOLUTION

**3** Soil forces (acting on the face, each layer with its own δ′): triangle · rectangle · triangle

$$P_s = \tfrac{1}{2}(3)(14.19) + 3(15.37) + \tfrac{1}{2}(3)(8.31) = 21.28 + 46.12 + 12.47 = 79.86\ \text{kN/m}$$

$$P_w = \tfrac{1}{2}(29.43)(3) = 44.14\ \text{kN/m}\ \text{(perpendicular to the wall)}$$

**4** Components, resultant and location

$$N = N_s + N_w = 77.04 + 44.14 = 121.18 \qquad T = 21.03\ \text{kN/m}$$

$$P = \sqrt{N^2 + T^2} = \mathbf{123.00}\ \text{kN/m} \qquad \beta = \tan^{-1}\frac{T}{N} = 9.84^\circ \qquad \bar{z} = \mathbf{1.69}\ \text{m}$$

The resultant is **123.00 kN/m** (soil 79.86 + water 44.14), 1.69 m above the base. The water force is 55% of that of the soil; at the boundary p jumps from 14.19 to 15.37 kPa (K changes).

Check the solution in GeoTerva ↗

# Problem statement

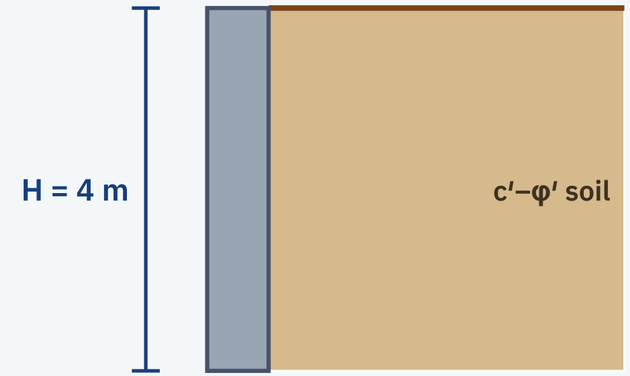
EXAMPLE

**REQUIRED**

**$K_a$**, tension depth $z_0$, **$P_a$** with and without a crack; comparison with Rankine.

**Given**

| | |
|---|---|
| Wall | $H = 4$ **m, vertical; horizontal fill** |
| Soil | **c′–φ′:** $\gamma = 18\ \mathrm{kN/m^3},\ \varphi' = 25^\circ,\ c' = 10$ **kPa;** $\delta' = 12.5^\circ = \varphi'/2$ |



# Solution (1/2)

SOLUTION

**1** Coulomb coefficient and cohesion term (adhesion is neglected)

$$K_a(25^\circ, 12.5^\circ) = \mathbf{0.3674} \qquad \sigma' = K_a\sigma'_v - 2c'\sqrt{K_a}$$

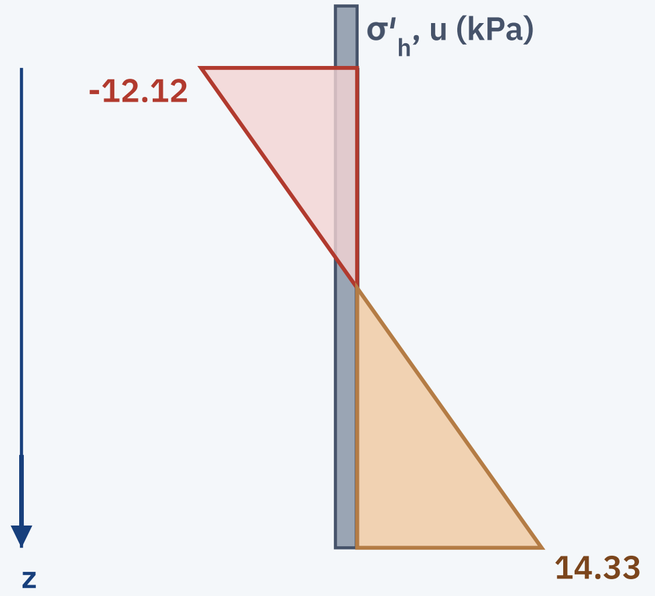
$$2c'\sqrt{K_a} = 2(10)(0.6061) = 12.12\ \text{kPa}$$

**2** At the top, at the base and at the tension depth

$$\sigma'(0) = -\mathbf{12.12}\ \text{kPa (tension)}$$

$$\sigma'(4) = 0.3674(18)(4) - 12.12 = \mathbf{14.33}\ \text{kPa}$$

$$z_0 = \frac{2c'}{\gamma\sqrt{K_a}} = \frac{20}{18(0.6061)} = \mathbf{1.83}\ \text{m}$$



# Solution (2/2)

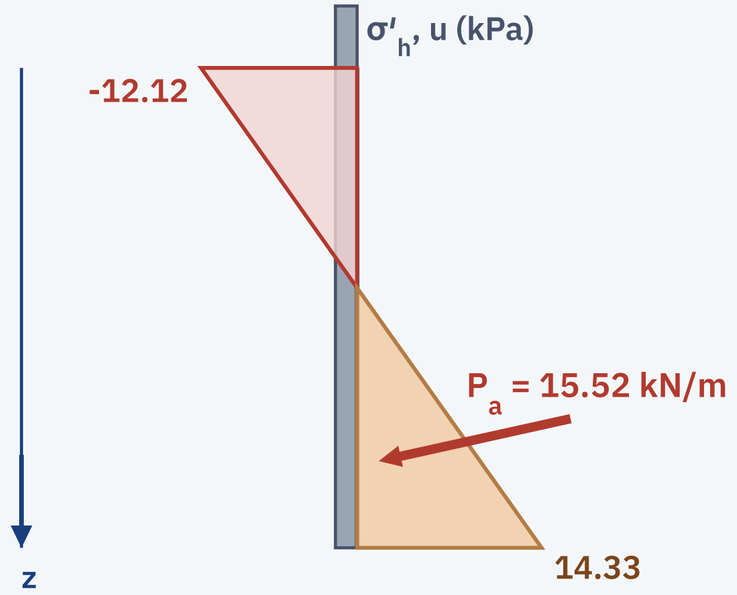
SOLUTION

**3** Cracked force (positive triangle only)

$$P_a = \tfrac{1}{2}(14.33)(4 - 1.83) = \mathbf{15.52}\ \mathrm{kN/m} \qquad \bar{z} = 0.72\ \mathrm{m}$$

$$\text{without crack: } P_a = 4.41\ \mathrm{kN/m}$$

Cracked $P_a$ = **15.52 kN/m**; uncracked (including the negative zone) 4.41 kN/m: if the tension zone is counted, the force comes out very small because it is small. Rankine (δ′ = 0) would give 18.59 kN/m.




Check the solution in GeoTerva ↗

# Scope of the Coulomb solution

THEORY

**The slip surface is planar.** In the active case this is a good approximation of the actual surface; in the passive case, for large δ′ it departs from the curved surface and the resistance is found to be **greater than it is** (caution for δ′ > φ′/3; logarithmic spiral module).

**Wall friction** δ′ is an input: it depends on the wall material and on the movement of the soil relative to the wall; rough estimates are δ′ = (1/2 – 2/3)φ′ (concrete–sand).

**Cohesive layer**: the cohesion term is taken in Rankine form with the layer's Coulomb K value; wall–soil adhesion is neglected; the tension zone is left out by means of a crack.

The **layered solution** is the engineering extension of the module (no closed-form layered Coulomb solution exists in the literature); for a single layer it reduces to the closed-form solution, and for δ′ = 0 and horizontal fill to Rankine.

**Earthquake** is in the Mononobe–Okabe module; Coulomb is the static solution.

Summary

# Key points

**Wedge**: W, F (at $\varphi'$ to its normal), P (at $\delta'$ to its normal); $P_a(\beta)$ by the sine rule; **active = largest** wedge, **passive = smallest** wedge

$\mathbf{K_a} = \cos^2(\varphi' - \theta) / \{\cos^2\theta \cos(\delta' + \theta)[1 + \sqrt{(\ldots)}]^2\}$; the force acts at H/3 from the base, at $\theta + \delta'$ to the horizontal; for $\delta' = 0$ and straight geometry, Rankine

**Wall friction** reduces the active force (up to $\delta' \approx 2\varphi'/3$) and increases the passive force rapidly; for large $\delta'$ in the passive case the planar wedge is unsafe

**Surcharge** $q_e = q \cos\theta / \cos(\theta - \alpha)$: $P = \tfrac{1}{2}K\gamma H^2\,(H/3) + K\,q_e\,H\,(H/2)$

**Layered + water**: $\sigma' = K_i\sigma'_v\cos\theta$, a jump at the boundary; u normal to the wall; $P = \sqrt{(N^2 + T^2)}$, $\beta = \tan^{-1}(T/N)$

**Cohesive layer**: $\sigma' = K\sigma'_v - 2c'\sqrt{K}$, the negative zone at a tension crack is disregarded

# References

Das, B.M. (2018). Principles of Geotechnical Engineering, 9th ed. Cengage Learning.

Coulomb, C.A. (1776). Essai sur une application des règles de Maximis et Minimis à quelques problèmes de statique relatifs à l'architecture. Mémoires de Mathématique et de Physique, Académie Royale des Sciences, Paris, 7, 343–382.

Das, B.M. (2016). Principles of Foundation Engineering, 8th ed. Cengage Learning — a surcharge on a sloping backfill.

**NOTE**

All numerical examples have been generated with the module's own calculation engine; the module gives the same values. The module opens with the wall of Example 3 (θ = 5°, α = 10°, δ′ = 15°).

GeoTerva

Online geotechnical analysis

# From theory to practice

Enter the slope of your own wall, the fill slope, its friction, its layers, its surcharge and the groundwater in GeoTerva; the module gives the force, its direction and its height of application together.

Open the module →

Theory page →

**Open on your phone**
The example modules need no registration.

If you would like to use them in your lectures: info@geoterva.com